Geometrical formalism of Quantum Mechanics and applications

Tensor Dynamics in Markovian open systems

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Contents

Introduction

The main goal of this work is the application of the geometrical formalism of Quantum Mechanics to the study of the dynamics of a particular kind of systems, known under the name of open systems. To the exposition of the fundamental characteristics of these we will devote the first chapter of this thesis. The study of this kind if systems is a matter of current importance due to the fact that it is in this framework that we can understand processes like decoherence, something that is necessary to advance in fields like quantum information and computation.
To carry out this task, we propose a different approach to the one usually employed, via an alternative mathematical formalism based on differential geometry. In the second chapter we expose the procedure to change fomalism and we show the direct application to the problem of open systems, by bringing the dynamic evolution from the space of density matrices to that of tensors. These tensors encode structures such as the matrix conmutator, what allows us to observe the transition from quantum to classical behaviour of a system under decoherence, when the non-conmutativity of some of its observables is lost.
The third chapter is devoted to the presentation of some examples of the new method proposed, and to the obtention of a result that establishes its range of applicability for a particular system and dynamics.

Chapter 1

Open systems

In this chapter we explain the basic concepts of open quantum systems, as they are known today. Our goal is to present the general form of the dynamics in one such system, under the additional simplifying hypothesis that it is Markovian, with the aim of applying this result in the central part of this work. Our main references along the whole chapter will be [20] and [2].

1.1 Time evolution in quantum systems

Let us consider a quantum system composed by two subsystems S and R, with associated state spaces $\mathcal{H}_S$ and $\mathcal{H}_R$. The state space of the global system is given by the tensor product of the spaces of the subsystems:

$$\mathcal{H} = \mathcal{H}_S \otimes \mathcal{H}_R$$

In what follows we will assume that $\dim \mathcal{H} < \infty$. Let us assume that this system is closed, i.e., it does not exchange information with any other system. In this case, the postulates of Quantum Mechanics tell us that the evolution of a state $|\psi\rangle \in \mathcal{H}$ is given by Schrödinger's equation:(taking $\hbar = 1$)

$$\frac{\partial|\psi(t)\rangle}{\partial t} = -iH(t)|\psi(t)\rangle$$

where $H(t)$ is the Hamiltonian operator. Thanks to the linearity of this equation we can define a **time evolution operator** $U(t, t_0)$ which, applied to a state $|\psi(t_0)\rangle$ produces the state at time t, $|\psi(t)\rangle = U(t, t_0)|\psi(t_0)\rangle$. From Schrödinger's equation follows as well that $U(t, t_0)$ is a unitary operator. When H is independent of time[1], we can give the following expression for the time evolution operator:

$$U(t - t_0) \equiv U(t, t_0) = e^{-i(t-t_0)H}$$

In many occasions we will be interested in working with mixed states, encoded by a**density matrix** ρ, which satisfies the following three conditions:

- It is Hermitian: $\rho^\dagger = \rho$
- It is positive definite: $\langle\psi|\rho|\psi\rangle \geq 0 \qquad \forall|\psi\rangle \in \mathcal{H}$

[1] If on the contrary H were dependent on t, the expression becomes somewhat more complicated since we have to include the time ordered exponential:

$$U(t, t_0) = \mathcal{T} e^{-i\int_{t_0}^{t} ds H(s)}$$

- Its trace equals one: $\operatorname{tr}\rho = 1$.

We shall denote by $D(\mathcal{H})$ the space of density matrices over $\mathcal{H}$. For example, for a two level system (qubit), any density matrix can be written in the form $\rho = \frac{1}{2}\left(\mathbb{1} + \sum_{i=1}^{3} x_i\sigma_i\right)$ with $\{\sigma_i\}_{i=1}^{3}$ the three Pauli matrices and $x_1^2+x_2^2+x_3^2 \leq 1$, since the purity of the state $P = \operatorname{tr}(\rho^2) = \frac{1}{2}(1 + x_1^2 + x_2^2 + x_3^2)$ must satisfy $P \leq 1$. $D(\mathcal{H})$ may thus be represented as a unit radius sphere in $\mathbb{R}^3$ called the *Bloch sphere*, given by $x_1^2 + x_2^2 + x_3^2 \leq 1$. For $P = 1$ we obtain the pure states (ρ of rank 1, surface of the sphere), and for $P < 1$ the mixed states (ρ of rank bigger than 1, interior of the sphere), up to the maximally mixed state $P = \frac{1}{2}$ ($\rho = \frac{1}{2}\mathbb{1}$, center of the sphere). It can be checked that the expected value of an observable in a state given by a density matrix ρ is given by

$$\langle A\rangle = \operatorname{tr}(\rho A)$$

Given an orthonormal basis $\{|\psi_k\rangle\}$ of $\mathcal{H}$, every density matrix adopts the form

$$\rho = \sum_k w_k |\psi_k\rangle\langle\psi_k|$$

with $w_k \geq 0, \sum_k w_k = 1$. From this expression it is easy to prove that the time evolution of ρ is given by von Neumann's equation:

$$\frac{\partial\rho(t)}{\partial t} = -i[H(t), \rho(t)] \implies \rho(t) = U(t,t_0)\rho(t_0)U^\dagger(t,t_0) \tag{1.1}$$

This is the usual frame in which Quantum Mechanics is formulated for closed or isolated systems. Nevertheless, in many ocassions we are interested in focusing only on the dynamics of one of the subsystems S, R. We may find that the dynamics of the global system is excessively complicated, as it happens if we take S as a physical system interacting with an environment R that is too complex or has too many degrees of freedom. In this kind of situations we may consider the reduced density matrix $\rho_S \in D(\mathcal{H}_S)$ induced by a global state $\rho \in D(\mathcal{H})$, which is given by the partial trace operation:

$$\rho_S = \operatorname{tr}_R \rho = \sum_k \langle\psi_k^R|\rho|\psi_k^R\rangle$$

with $\{\psi_k^R\}$ a Hilbert basis of the space $\mathcal{H}_R$. This way we are able to perform measurements, and in general to work on S directly, without having to consider the global state of the system. For example, if A is an observable in $\mathcal{H}_S$, its expected value in the state ρ will be

$$\langle A\rangle = \operatorname{tr}(A\rho_S) = \operatorname{tr}\{(A \otimes I)\rho\}$$

The dynamics of this ρ_S will now take the following form

$$\rho_S(t) = \operatorname{tr}_R\{U(t,t_0)\rho(t_0)U^\dagger(t,t_0)\} \tag{1.2}$$

and if subsystems S and R interact with each other, in general we cannot obtain from (1.2) a unitary evolution[2] like (1.1) for ρ_S. We thus face the time evolution of an **open**, which does not obey the usual rules, since, for example, it allows the rank of the density matrix to grow, and in general allows the purity P of our system to decrease with time (decoherence), when both were

[2] This would indeed be possible if the evolution operator factorized as $U(t,t_0) = U_S(t,t_0) \otimes U_R(t,t_0)$, but in general that will not be the case.

invariant under unitary evolution.

1.2 Universal dynamical maps

The equation (1.2) gives us the time evolution of a reduced density matrix. Hence we would like to write it as a dynamical map that allows us to evolve a certain state $\rho \in D(\mathcal{H}_S)$ from some time t_0 to some other time t, in the following manner:

$$\mathcal{E}_{(t,t_0)}\left(\rho(t_0)\right) = \rho(t)$$

Of course, not all possible dynamical maps $\mathcal{E}$ are admissible as true physical evolutions. Those which are receive the name of **universal dynamical maps**[3], o UDM por sus siglas en inglés [20]. The adjective *universal* means in this case that these maps can be defined independently of the density matrix they are acting on, a very desirable property if they are to represent a physical evolution. In what follows we will see what kind of maps UDMs are.

A first characterization we can give for UDMs is the following: let us fix a state $\rho_R \in D(\mathcal{H}_R)$. This can be a reference state of the system, for example a thermal state. Once ρ_R has been chosen, we build the separable state[4] $\rho_S(t_0) \otimes \rho_R$ for any initial density matrix $\rho_S(t_0)$ and we define

$$\mathcal{E}_{(t,t_0)}[\rho_S(t_0)] = \mathrm{tr}_R \left\{ U(t,t_0)[\rho_S(t_0) \otimes \rho_R] U^\dagger(t,t_0) \right\} \tag{1.3}$$

An evolution of this kind can be written only in terms of operators acting on density matrices of $D(\mathcal{H}_S)$[2]. It is enough to use the spectral decomposition of our reference state:

$$\rho_R = \sum_n \lambda_n |\phi_n\rangle\langle\phi_n| \tag{1.4}$$

and combining (1.3) and (1.4) we obtain:

$$\mathcal{E}_{(t,t_0)}[\rho_S(t_0)] = \sum_{nm} K_{nm}(t,t_0)\rho_S(t_0)K_{nm}(t,t_0)^\dagger \tag{1.5}$$

where

$$K_{nm}(t,t_0) = \sqrt{\lambda_m}\, \mathrm{tr}_R \left\{ |\phi_m\rangle\langle\phi_n| U(t,t_0) \right\}$$

Merging both indices in one, $n, m \to \alpha$, (note that at most that index will have to take N^2 values, where N is the dimension of the space R) we conclude that any UDM can we written in the following general form, which is handier in operational terms (**Kraus representation**):

$$\rho_S(t) = \sum_\alpha K_\alpha(t,t_0)\rho_S(t_0)K_\alpha(t,t_0)^\dagger \qquad \text{con} \qquad \sum_\alpha K_\alpha^\dagger(t,t_0)K_\alpha(t,t_0) = \mathbb{1} \tag{1.6}$$

We can check on this general form that the result of evolving a density matrix with a UDM is itself a density matrix, an unavoidable requirement if we want it to have a physical meaning. Indeed we have:

[3] Depending on the reference, the reader might find other names such as *quantum dynamical map* or *quantum operation*, this last one especially in the context of quantum computation.

[4] It is reasonable to assume that, when the system has been prepared, it begins its evolution in a separable state.

- $\rho(t)$ is Hermitian:

$$\rho(t)^\dagger = \sum_\alpha (K_\alpha(t,t_0)\rho(t_0)K_\alpha(t,t_0)^\dagger)^\dagger = \sum_\alpha K_\alpha(t,t_0)\rho(t_0)K_\alpha(t,t_0)^\dagger = \rho(t)$$

- $\rho(t)$ is positive definite, since so is every term in the sum: $K_\alpha(t,t_0)\rho(t_0)K_\alpha(t,t_0)^\dagger$

$$\langle\psi|K_\alpha(t,t_0)\rho(t_0)K_\alpha(t,t_0)^\dagger|\psi\rangle = (K_\alpha(t,t_0)^\dagger|\psi\rangle)^\dagger\rho(t_0)(K_\alpha(t,t_0)^\dagger|\psi\rangle \geq 0 \qquad \forall|\psi\rangle \in \mathcal{H}$$

- And its trace equals one thanks to the condition on the operators $K_\alpha(t,t_0)$:

$$\begin{aligned}\mathrm{tr}\,\rho(t) = \mathrm{tr}\left(\sum_\alpha K_\alpha(t,t_0)\rho(t_0)K_\alpha(t,t_0)^\dagger\right) = \sum_\alpha \mathrm{tr}\,(K_\alpha(t,t_0)\rho(t_0)K_\alpha(t,t_0)^\dagger) = \\ = \mathrm{tr}\left\{\left(\sum_\alpha K_\alpha(t,t_0)^\dagger K_\alpha(t,t_0)\right)\rho(t_0)\right\} = \mathrm{tr}\,\rho(t_0) = 1\end{aligned}$$

1.2.1 Completely positive maps

Definición 1.1. *A linear map $F : V \longmapsto V$ is* ***completely positive*** *if*

$$F \otimes \mathbb{1} : V \otimes W \longmapsto V \otimes W$$

is positive independently of the space W, and in particular of its dimension[5]*.*

A very relevant characterization of UDMs is the following [20]:

> *A UDM is a completely positive, trace-preserving linear map.*

Let us see what it means for a map to be *completely positive* and why it is important (in fact indispensable) from a physical point of view. Let us assume that we have a third subsystem W, in addition to the other two we had been dealing with. Let us as welll assume that this subsystem does not interact with the other, so that it follows its own unitary evolution $U_W(t,t_0)$ in a totally independent way to the other subsytems. We focus on the subsystem SW, which we assume starts from an initial state $\rho_{SW}(t_0)$. Its dynamics will be given, since S and W are independent, by the tensor product of the dynamics of the two subsystems: the unitary evolution of W, $\mathcal{U}^W_{(t,t_0)}$ and the more general UDM of S:

$$\rho_{SW}(t) = \mathcal{E}_{(t,t_0)} \otimes \mathcal{U}^W_{(t,t_0)}[\rho_{SW}(t_0)]$$

Since in the end what we just gave isn't but the reduced dynamics of a subsystem SW in interaction with another part of the global system, R, even though only through S, its evolution must be given by a UDM, so that we conclude

$$\mathcal{E}_{(t,t_0)} \text{ is a UDM } \implies \mathcal{E}_{(t,t_0)} \otimes \mathcal{U}^W_{(t,t_0)} \text{ is a UDM}$$

[5] By specifying the dimension of W one defines the concept of n-positivity: F is n-positive when $F \otimes \mathbb{1}_n$ is positive. In finite dimension, F is completely positive if it is n-positive for all n.

where $U^W_{(t,t_0)}$ is a unitary evolution. In particular, hence, $\mathcal{E}_{(t,t_0)} \otimes \mathcal{U}^W_{(t,t_0)}$ must preserve the positivity of density matrices. Since we can factor

$$\mathcal{E}_{(t,t_0)} \otimes \mathcal{U}^W_{(t,t_0)} = (\mathcal{E}_{(t,t_0)} \otimes \mathbb{1})(\mathbb{1} \otimes \mathcal{U}^W_{(t,t_0)})$$

and the factor $(\mathbb{1} \otimes \mathcal{U}^W_{(t,t_0)})$ is unitary (and hence positive), the condition is imposed on the other factor, and it remains that $\mathcal{E}_{(t,t_0)}$ is completely positive. It can be shown that the condition of being completely positive is stronger than the one of only being positive, i.e. not every positive map is completely positive[6]. The theorem of representation for completely positive maps, which connects this characterization with the one we gave previously was proved by Karl Kraus [15].

1.3 Markovianity and semigroups

The dynamics of open systems would not be as interesting if it did not possess certain characteristics that distinguish it drastically from the evolution in closed systems. Of them one of the most relevant is that, in general, a UDM will not be reversible. Let us remember that, in a closed system, the evolution operator family acquires a group structure, where every element is invertible. In the simplest case, when H no depende del tiempodoes not depend on time, $U(t) = e^{-iHt}$ has $U(-t) = e^{iHt}$ as an inverse. The situation is nevertheless quite different for open systems. Given a UDM $\mathcal{E}_{(t_0,t)}$, we may ask ourselves whether there is another UDM that acts as its inverse, as we saw it happens for unitary evolutions:

$$\mathcal{E}_{(t_0,t)} = \mathcal{E}^{-1}_{(t,t_0)}$$

The answer to this question is usually negative and is given by the following theorem whose proof we can see in [20]:

Teorema. *A UDM has an inverse UDM if and only if it is unitary.*

Thus, open systems lose the reversibility property as soon as their dynamics is no longer unitary. This implies that a family of operators $\mathcal{E}_{(t,s)}$ will no longer be able to give rise to a group, but at most to a semigroup or an evoution family, as we will soon see.

1.3.1 Markovian evolution

In not very technical terms, Markovian evolution is described as that which has no "memory", that is to say, that which is only affected by the current state of the system, and not by the whole evolution history of it. In the caso of a UDM, we will say it is Markovian if it admits the following composition law:

$$\mathcal{E}_{(t,t_0)} = \mathcal{E}_{(t,t_1)}\mathcal{E}_{(t_1,t_0)} \tag{1.7}$$

for any intermediate time t_1. The lack of memory is reflected in the differential equation for the density matrix being first order:

$$\frac{d\rho(t)}{dt} = \lim_{h\to 0} \frac{\rho(t+h) - \rho(t)}{h} = \lim_{h\to 0} \frac{(\mathcal{E}_{(t+h,t)} - I)\rho(t)}{h} = L(t)\rho(t) \tag{1.8}$$

[6]The interested reader can find an example (transposition of a qubit) in [19].

where we have used the composition law (1.7). $L(t)$ is the generator of the evolution, also known as *Lindblad superoperator*. This is, of course, an approximation. In general, the UDMs do not have to satisfy (1.7), because $\mathcal{E}_{(t,t_1)}$ might not be a UDM. We might think, for instance, about defining it as $\mathcal{E}_{(t,t_1)} = \mathcal{E}_{(t,t_0)}\mathcal{E}^{-1}_{(t_1,t_0)}$ but we have already seen that, in general, even if an UDM is bijective, and hence has an inverse, this need not be a UDM, so we cannot go on that way. The suposition of Markovianity is therefore a simplifying hypothesis, since it is conditioned to the decay time of system-environment correlations to be much smaller that the relaxation time of the system, to be able to neglect memory effects.
From now on we will assume that the evolution of our system is Markovian. The evolution operators form then an **evolution family**, characterized by

$$\mathcal{E}_{(s,s)} = \mathbb{1} \qquad \mathcal{E}_{(t,s)} = \mathcal{E}_{(t,r)}\mathcal{E}_{(r,s)} \qquad \text{si } t \geq r \geq s$$

Or in the case in which $\mathcal{E}_{(t,s)} \equiv \mathcal{E}_{t-s}$, a **dynamical semigroup**[7]:

$$\mathcal{E}_0 = \mathbb{1} \qquad \mathcal{E}_t\mathcal{E}_s = \mathcal{E}_{t+s} \qquad t, s \geq 0$$

This kinf of structures present interesting properties. Let us start by semigroups. Since we will always assume that the evolution of our system is sufficiently smooth, it is worth for us to restrict ourselves to uniformly continuous semigroups, i.e., those which satisfy[8]

$$\|\mathcal{E}_t - \mathcal{E}_s\| \to 0 \qquad \text{cuando } t \to s$$

The advantage of this semigroups is that automatically the map $t \to \mathcal{E}_t$ is differentiable and we may characterize the semigroup by a **generator** L, satisfying

$$\frac{d\mathcal{E}_t}{dt} = L\mathcal{E}_t$$

L is a linear operator over the space where the evolution operators live. This generator is the same L that we saw in (1.8), in the particular case in which it does not depend on time. If the semigroup is **contractive**,i. e.,

$$\|\mathcal{E}_t\| \leq 1 \qquad \forall t \geq 0$$

the exponential of the operator L indeed generates the whole semigroup:

$$\mathcal{E}_t = e^{tL}\mathcal{E}_0 = e^{tL}\mathbb{1}$$

All the semigroups with which we will work will have to be contractive so that the image of a density matrix is a density matrix. This is due to the following property, whose proof we find in [20]:

> *A linear map $\mathcal{E}$ over the set of trace class operators over a Hilbert space leaves the set of density matrices invariant if and only if it is contractive and trace preserving.*

The requirement that a semigroup is contractive can be translated into a series of conditions on its generator (Hille-Yosida and Lumer-Phillips theorems). We omit the discussion of this

[7] A semigroup, contrary to a group, does not require the existence of inverses.
[8] The norm that appears in this equation is the norm induced on the space of linear operators over a Banach space by the norm of this Banach space $\|T\| = \sup_{\|x\|=1} \|T(x)\|$.

conditions since soon we will present string requirements for L.
Evolution families can be given a similar though slightly more complicated treatment than semigroups. In the case of being differentiable they also have a generator $L(t)$, which depends this time on the parameter t, since otherwise the family is reduced to a semigroup like the previous ones. From now on we will deal with the case where the evolution is given by a semigroup, and we will care about the generator L which characterizes it.

1.3.2 General form of the generator of a totally positive dynamical semigroup

We finish this first chapter where we have made an introduction to the peculiarities of time evolution in open systems with an important result: the characterization of the generator of a totally positive dynamical semigroup. Because of everything we have exposed above, it will be this kind of semigroups that will give us the evolution in open quantum systems under the Markovian approximation. In 1976, Lindblad published which was this general form [16] basing himself in previous work by Kossakowski [14]. Soon after Gorini, Kossakowski y Sudarshan [10], working independently from Lindblad, proved that in the general case of systems a finite number N of levels, the most general form of the generator of a totally postive dynamical semigroup is

$$L\rho = -i[H,\rho] + \frac{1}{2}\sum_{i,j=1}^{N^2-1} c_{ij}([F_i, \rho F_j^\dagger] + [F_i\rho, F_j^\dagger]) \tag{1.9}$$

where H is a Hermitian operator, $\{F_i\}$ is a set of $N^2 - 1$ operators such that together with the identity they form a basis of the space of complex $N \times N$ matrices which is orthogonal with respect to the scalar product $(F_i, F_j) = \mathrm{tr}(F_i^* F_j)$, and (c_{ij}) is a complex positive definite matrix. The first term is known as the *Hamiltonian* part, whereas the rest is called *dissipative* part. We can choose H to be traceless, then it is unique for a fixed L,and so are the coefficients c_{ij} once we fix the F_i of the basis. We must be cautious to observe that, in general, H will not be equal to the Hamiltonian of the system considered as a closed system.

Chapter 2

Geometrical formalism

In this chapter we describe a mathematical formalism that is different to the one usually employed to deal with quantum systems, and we are going to apply it to the problem of studying the dynamics of open quantum systems. We will start by characterizing the algebraic structures defined over the set of Hermitian operators over $\mathcal{H}$ to turn them afterwards into geometric structures over its dual. Finally we will show how we can interpret the dynamics as an evolution over these structures.

2.1 Mathematical structure of $\mathrm{Herm}\,\mathcal{H}$

We start from the Hilbert space, $\mathcal{H}$ which contains the possible states of our system, and we assume $\dim \mathcal{H} = N < \infty$. The set of linear operators acting on $\mathcal{H}$ has a particular structure:

Definición 2.1. *A **C*-algebra** $(\mathcal{A}, \cdot, \|\cdot\|, ^*)$ is a complex vector state endowed with*

- *a **product** $\cdot$, which gives it a structure of linear associative algebra*
- *a **norm** $\|\cdot\|$, which gives it a structure of Banach space (complete normed space) and such that the product is continuous (i.e. submultiplicativity is satisfied $\|AB\| \leq \|A\|\|B\|$)*
- *an **involution**, i.e., a map $^* : \mathcal{A} \to \mathcal{A}$ such that*

$$(A + \lambda B)^* = A^* + \bar{\lambda} B^* \qquad (AB)^* = B^* A^* \qquad (A^*)^* = A$$

and the so-called C^* identity *is satisfied:* $\qquad \|AA^*\| = \|A\|^2$

Indeed, for a finite number N of levels, $\mathcal{H} \cong \mathbb{C}^N$ and the set of linear operators acting on it is $M(N)$ ($N \times N$ matrices with complex entries). It can be seen that, with the usual matrix product (given by the composition of linear maps), the norm

$$\|A\| = \sup_{0 \neq x \in \mathcal{H}} \frac{\|Ax\|}{\|x\|}$$

and the involution given by Hermitian conjugation $A^* = A^\dagger$, $M(N, \mathbb{C})$ acquires a C*-algebra structure. In its more general formulation, a quantum system is described by its C*-algebra of operators. The GNS (Gelfand-Naimark-Segal) theorem allows to obtain from it the states as linear positive normalized functionals over the elements of the C*-algebra and proves that the set of this states is precisely a Hilbert space [21][22][23].

The Hermitian operators over $\mathcal{H}$, which represent the observables of the system, play a major role inside the C*-algebra since they conform its *real part*, i.e., the set of operators that are invariant under the involution, $\text{Herm}\,\mathcal{H} = \{A \in M(N, \mathbb{C}) | A^\dagger = A\}$. In general the real part of a C*-algebra has a structure known as Lie-Jordan-Banach algebra, or LJB algebra. Showing explicitly this structure is the main goal of this section, and we will proceed part by part, so we beg the reader for patience.
Inside our C*-algebra $M(N)$ we can define, from the usual matrix product[1] a new operation $\circ$ known as Jordan product or, more commonly, anticonmutator, which is the symmetric part of the associative product, $A \circ B := \frac{1}{2}(AB + BA)$. With this operation $M(N)$ turns into a Jordan algebra:

Definición 2.2. *A **Jordan algebra** $(\mathcal{A}, \circ)$ is a vector space $\mathcal{A}$ endowed with a conmutative bilinear product $\circ : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$ such that*

$$(a^2 \circ b) \circ a = a^2 \circ (b \circ a) \qquad \forall a, b \in \mathcal{A}$$

In the same way, we can define another new operation from the antisymmetric part of the associative product, which is the one know as Lie bracket, or more commonly, anticonmutator, $[A, B] := AB - BA$. With this new operation, $M(N)$ acquires a Lie algebra structure:

Definición 2.3. *A **Lie algebra** $(\mathcal{A}, [,])$ is a vector space $\mathcal{A}$ endowed with a bilinear product $[\cdot, \cdot] : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$ which:*

- *Is antisymmetric:* $[a, b] + [b, a] = 0 \qquad \forall a, b \in \mathcal{A}$
- *Satisfies the Jacobi identity* $\quad [a, [b, c]] + [b, [c, a]] + [c, [a, b]] = 0 \qquad \forall a, b, c \in \mathcal{A}$

Let us stop for a moment on Lie algebras. This kind of structures play a very important role in Mathematics, and especially in Physics, due to the fact that there is always one of them associated to every Lie group:

Definición 2.4. *A **Lie group** is a group endowed with a structure of differentiable manifold such that the group's product and inversion operations are differentiable maps.*

Physics is full of Lie groups which encode the symmetries of a system, such as the spacetiem translation group, or the rotation group.
$M(N, \mathbb{C})$ is a real manifold of dimension $2N^2$, which contains an open submanifold of capital importance, the general linear group $GL(N, \mathbb{C})$, or the group of invertible matrices. The general linear group is hence a Lie group. Its associated Lie algebra is denoted $\mathfrak{gl}(N, \mathbb{C})$ and turns out to be isomorphic to $M(N, \mathbb{C})$. Also, the corresponding Lie bracket is precisely the usual conmutatorof matrices, hence we recover waht we already knew: $M(N, \mathbb{C})$ is a Lie algebra.
The elements of a basis of a finite dimensional Lie algebra $\mathfrak{g}$ are the generators of the associated Lie group G via the **exponential map** $\exp : \mathfrak{g} \to G$ something we will make use of soon[2].

[1] From now on we will call it *associative product*, sine we are going to define new operations that will not satisfy associativity.

[2] The exponential map from any Lie algebra to its associated Lie group always exists. In the particular case of $GL(n, \mathbb{C})$, the exponential map coincides with the usual matrix exponential, defined by the convergent series

$$e^A = \sum_{n=0}^{\infty} \frac{A^n}{n!}$$

Once we have seen all this, we are ready to define what a Lie-Jordan-Banach algebra is:

Definición 2.5. *A **Lie-Jordan-Banach algebra** $(\mathcal{A}, \circ, [,], \|\cdot\|)$ is an algebra endowed with a Jordan product $\circ$ and a Lie bracket $[,]$ which satisfy the following compatibility conditions:*

- *The Lie brackt defines a derivation of the Jordan product, i.e., the Leibniz rule is satisfied:*

$$[a, b \circ c] = [a, b] \circ c + b \circ [a, c] \qquad \forall a, b, c \in \mathcal{A}$$

- *The associators of both products are proportional, i.e., for some $\hbar \in \mathbb{R}$:*

$$(a \circ b) \circ c - a \circ (b \circ c) = \hbar^2 [[a, b], c] - [a, [b, c]] \qquad \forall a, b, c \in \mathcal{A}$$

and with a norm $\|\cdot\|$ so that it has a structure of Banach space and satisfies:

$$\|a \circ b\| \leq \|a\|\|b\| \qquad \|[a, b]\| \leq \frac{1}{|\hbar|}\|a\|\|b\| \qquad \|a^2\| = \|a\|^2 \qquad \|a^2\| \leq \|a^2 + b^2\|$$

for all $a, b \in \mathcal{A}$.

Therefore, to endow $\mathrm{Herm}\,\mathcal{H}$ with an LJB structure we must start by seeing that it is a Lie algebra and a Jordan algebra. But this last part is very easy, since we can restrict the Jordan product of $M(N, \mathbb{C})$ to $\mathrm{Herm}\,\mathcal{H}$ and we note that the operation is closed:

$$A, B \in \mathrm{Herm}\,\mathcal{H} \implies \frac{1}{2}(AB + BA) \in \mathrm{Herm}\,\mathcal{H}$$

However, the Lie bracket of $M(N, \mathbb{C})$ cannot be used as a Lie bracket for the Hermitian operators, since its restriction to this set turns out not to be a closed operation:

$$A, B \in \mathrm{Herm}\,\mathcal{H} \not\Longrightarrow (AB - BA) \in \mathrm{Herm}\,\mathcal{H}$$

Hence we have to find a new Lie bracket $[,]_-$ closed in the Hermitian operators. It is not difficult to find an *ad hoc* one which satisfies this property, but we shall obtain it in a more reasoned way that may provide us as well with new information about the sets we deal with.
Inside $GL(N, \mathbb{C})$ we find the **unitary group** $\mathcal{U}(N)$, which is the set of operators U which preserve the hermitian structure of the Hilbert space $\langle U\psi|U\chi\rangle = \langle\psi|\chi\rangle$ and is a Lie subgroup of $GL(N, \mathbb{C})$. The Lie algebra associated to the unitary group $\mathcal{U}(N)$ will hence be a subalgebra of $\mathfrak{gl}(n, \mathbb{C})$ which we denote $\mathfrak{u}(N)$ or $\mathfrak{u}$ to abbreviate. By using the exponential map we can identify the elements of $\mathfrak{u}$ as the antihermitian operators:

$$e^{tT} \in U \implies (e^{tT})^\dagger = (e^{tT})^{-1} \implies T^\dagger = -T$$

The set of Hermitian operators can then be written as $\mathrm{Herm}\,\mathcal{H} \cong i\mathfrak{u}$, by defining the map $\phi : \mathfrak{u} \longmapsto i\mathfrak{u}, \phi(A) = iA$ which relates both sets. ϕ allows us to transport the Lie algebra structure of $\mathfrak{u}$ to $i\mathfrak{u}$, hence resulting the Lie bracket:

$$[A, B]_- = \phi([\phi^{-1}(A), \phi^{-1}(B)]) = -i[A, B] \qquad A, B \in i\mathfrak{u}$$

Even more, if it were necessary we could use it to endow $\mathfrak{u}$ with a Jordan product, since again the restriction of the one defined in $M(N, \mathbb{C})$ is not valid since it is not closed in $\mathfrak{u}$. Hence

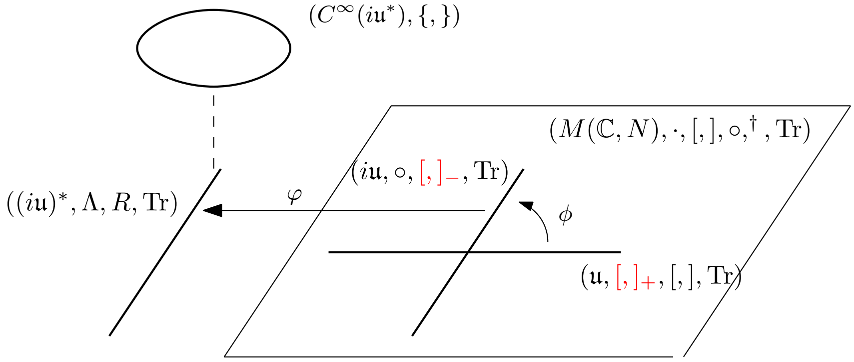

Figure 2.1: Mathematical strutures described in this section (in red the ones transported by the isomorphism ϕ).

transporting the one in $i\mathfrak{u}$ by ϕ we obtain a Jordan product $[,]_+$

$$[A, B]_+ = \phi^{-1}(\phi(A) \circ \phi(B)) = \frac{i}{2}(AB + BA) \qquad A, B \in \mathfrak{u}$$

In general, the isomorphism ϕ existent between both spaces allows us to use either of them to describe the physics of the system. The advantage of choosing $i\mathfrak{u}$ is that its elements have real eigenvalues, what makes its interpretation as possible results of a physical measurement more straightforward.

To finish, both $M(N, \mathbb{C})$ and its subspaces $\mathfrak{u}$ and $i\mathfrak{u}$ are endowed with a metric or scalar product via the trace $(A, B) = \mathrm{tr}\, A^\dagger B$, what makes them automatically normed spaces with the Frobenius norm $\|A\| = \sqrt{\mathrm{tr}\, A^\dagger A}$. From here it is not difficult to check that $(i\mathfrak{u}, \circ, [,]_-, \|\cdot\|)$ is an LJB algebra. It is this structure which contains all the physics of the system. The relation between C*-algebras and LJB algebras goes indeed beyond this, since it can be proved that the complexification of a given LJB algebra is the only C*-star algebra that has it as its real part, i.e., the observables determine the set of operators [9]. The reader will find a graphical summary of this section and the next in figure 2.1.

2.2 Geometry over $(i\mathfrak{u})^*$

In this section we are going to turn the algebraic structure with which $i\mathfrak{u}$ is endowed into a geometric structure over its dual $(i\mathfrak{u})^*$. This will be possible thanks to the fact that, in finite dimension, by Riesz theorem the two spaces are identified by an isomorphism given by the scalar product, $\varphi : i\mathfrak{u} \longmapsto (i\mathfrak{u})^*, A \longmapsto (A, \cdot)$. This identification allows us to transport any structure that we need between both spaces[3], in the same way that we did with $i\mathfrak{u}$ and $\mathfrak{u}$. Since $i\mathfrak{u}$ is a Lie algebra, its dual has what we know as a canonical Poisson structure. Let us see how this is. In the first place, $(i\mathfrak{u})^*$ is a real vector space, so that by choosing a basis $\{e_i\}$ we can endow it with a structure of differentiable manifold with a unique global chart. Over this manifold we have the set of infinitely differentiable functions $C^\infty(i\mathfrak{u}^*)$, where we find, among others, the elements of the bidual $i\mathfrak{u}^{**}$, that is to say, the linear functionals over $(i\mathfrak{u})^*$. Since we are in finite dimension, we can identify the bidual with the original space $A \in i\mathfrak{u} \longleftrightarrow \hat{A} \in (i\mathfrak{u})^{**}$ in a way

[3] We must warn that this has led to the use of formalisms like the one we are to describe in several different spaces, all of them isomorphic, and thus equivalent, hence in the bibliography we can find these same constructions realized over, for example, $\mathfrak{u}^*$, the dual of $\mathfrak{u}$.

such that[4] $\hat{A}(\xi) = (\xi, A) \quad \forall \xi \in (i\mathfrak{u})^*$. It is then perfectly sensible to define the following tensors (bivectors) over $(i\mathfrak{u})^*$:

$$\Lambda_\xi(d\hat{A}, d\hat{B}) = (\xi, [A,B]_-) \qquad R_\xi(d\hat{A}, d\hat{B}) = (\xi, A \circ B)$$

We can define this tensors in terms of the basis of bivectors $\frac{\partial}{\partial x_i} \otimes \frac{\partial}{\partial x_j}$ associated to the basis $\{e_i\}$ if we previously define the structure constants:

$$[e_i, e_j]_- = \sum_k c_{ij}^k e_k \qquad e_i \circ e_j = \sum_k d_{ij}^k e_k$$

The component of the Λ tensor in $\frac{\partial}{\partial x_i} \otimes \frac{\partial}{\partial x_j}$ will hence be given by

$$\Lambda_\xi(d\hat{e}_i, d\hat{e}_j) = (\xi, [e_i, e_j]_-) = (\xi, \sum_k c_{ij}^k e_k) = \sum_k c_{ij}^k x_k(\xi)$$

where we denote by x_k the coordinate function $(\cdot, e_k) = \hat{e}_k$. In the same way we act with the tensor R, so that both are expressed as

$$\Lambda = \sum c_{ij}^k x_k \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j} \qquad R = \sum d_{ij}^k x_k \frac{\partial}{\partial x_i} \otimes_S \frac{\partial}{\partial x_j}$$

where $\wedge$ and $\otimes_S$ denote the antisymmetrized and symmetrized tensor products respectively:

$$\frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y} = \frac{\partial}{\partial x} \otimes \frac{\partial}{\partial y} - \frac{\partial}{\partial y} \otimes \frac{\partial}{\partial x} \qquad \frac{\partial}{\partial x} \otimes_S \frac{\partial}{\partial y} = \frac{\partial}{\partial x} \otimes \frac{\partial}{\partial y} + \frac{\partial}{\partial y} \otimes \frac{\partial}{\partial x}$$

The tensor Λ is antisymmetric, it is obtained canonically from the Lie algebra structure in $i\mathfrak{u}$ and it is called the Poisson tensor. Mathematically, Poisson tensors are a particular class of antisymmetric bivectors characterized by $[\Lambda, \Lambda]_S = 0$ where $[,]_S$ denotes the Schouten bracket, a generalization of the vector field conmutator for multivectors of arbitrary degree. In Physics they are important since they induce, on the space of functions C^∞ over the manifold where they are defined, a bilinear operation known as Poisson bracket, $\{f, g\} = \Lambda(df, dg)$, which is used in Classical Mechanics to define the dynamics. The tensor R is a symmetric tensor induced by the Jordan algebra structure of $i\mathfrak{u}$.

Let us see which role these tensors play in the dynamics. If we have an evolution given by a first order linear differential equation, we can translate it in geometrical terms as follows. Assume we have the differential equation for a curve in $(i\mathfrak{u})^*$, $\dot{\gamma}(t) = K\gamma(t)$ with K a certain linear operator over $(i\mathfrak{u})^*$, that is, $K \in \mathfrak{gl}((i\mathfrak{u})^*)$. We want to rewrite it as

$$\dot{\gamma}(t) = X^K_{\gamma(t)}$$

with $X^K \in \mathfrak{X}((i\mathfrak{u})^*)$ a vector field over $(i\mathfrak{u})^*$. If we decompose both equations in components respect to a basis in $(i\mathfrak{u})^*$, which can be the dual basis to the one we have in $i\mathfrak{u}$, $\tilde{e}_i = \varphi(e_i)$, we have $\gamma(t) = \sum \gamma_j(t)\tilde{e}_j$ y

$$\dot{\gamma}_j = (K\gamma(t), e_j) = (\gamma(t), K^\dagger e_j) \qquad \dot{\gamma}_j = X^K_{\gamma(t)}(d\hat{e}_j)$$

[4]Here usual notation is a bit confusing: we usually denote in the same way the scalar product of two matrices A, B of $i\mathfrak{u}$, (A, B), and the result of applying $\xi \in (i\mathfrak{u})^*$ to $A \in i\mathfrak{u}$, (ξ, A).

where $K^\dagger$ is the adjoint operator of K. By comparison we can deduce how the field we look for acts on a function $\hat{A}$:

$$X_\xi^K(\hat{A}) = (K\xi, A) = (\xi, K^\dagger A)$$

The Poisson tensor allows us to define a Hamiltonian dynamics on $(i\mathfrak{u})^*$. Let us assume for example that we are interested in translating to this formalism the equation that governs the time evolution of the density matrices, which is von Neumann's equation $\dot{\rho} = \mathrm{ad}_H(\rho)$ with $\mathrm{ad}_H = [H, \cdot]_-$ in the role of the generic operator [5] K. This way we compute the operator $\mathrm{ad}_H^\dagger$:

$$(\mathrm{ad}_H(A), B) = \mathrm{tr}\left((-i(HA - AH))^\dagger B\right) = i\,(\mathrm{tr}(AHB) - \mathrm{tr}(HAB)) = i\,(\mathrm{tr}(AHB) - \mathrm{tr}(ABH)) =$$
$$= \mathrm{tr}\,(Ai(HB - BH)) = (A, -\mathrm{ad}_H(B)) \implies \mathrm{ad}_H^\dagger = -\mathrm{ad}_H$$

and we conclude that the field that gives us the Hamiltonian dynamics for the density matrices is

$$X_\xi^H(\hat{B}) = (\xi, -\mathrm{ad}_H(B)) = -\Lambda_\xi(d\hat{H}, d\hat{B}) \implies X^H = -\Lambda(d\hat{H}, \cdot)$$

2.3 Evolution in the space of tensors

We have seen in chapter 1 that, in general, in open systems the evolution of the density matrix is not given by von Neumann's equation but by a more general Lindblad operator (1.9). This field has a more complete interpretation in geometric terms. To see it, we proceed to diagonalize the matrix c_{ij} by makin a change of basis in the F_i. Since it is positive definite, the eigenvalues of the matrix are positive, and a small computation leads us to express (1.9) as

$$L\rho = -i[H,\rho] + \frac{1}{2}\sum_{i=1}^{N^2-1}([K_i, \rho K_i^\dagger] + [K_i\rho, K_i^\dagger]) = -i[H,\rho] + \frac{1}{2}\sum_{i=1}^{N^2-1}(2K_i\rho K_i^\dagger - K_i^\dagger K_i\rho - \rho K_i^\dagger K_i) =$$
$$= -i[H,\rho] + J\circ\rho + \sum_{i=1}^{N^2-1} K_i\rho K_i^\dagger$$

where $J = \sum_i K_i^\dagger K_i$. The operators K_i receive the name of **Kraus operators**. This way, the field has the form

$$\frac{d\rho}{dt} = L\rho = [H,\rho]_- + J\circ\rho + \sum_\alpha K_i\rho K_i^\dagger \tag{2.1}$$

or expressing it as a vector field, like we saw in the previous section:

$$\frac{d\rho}{dt} = X^L = -\Lambda(d\hat{H}, \cdot) + R(d\hat{J}, \cdot) + X_K$$

Hence, the field whose integral curves give the evolution of the density matrix is made of a Hamiltonian field $X_H = -\Lambda(d\hat{H}, \cdot)$, a gradient field $X_J = R(d\hat{J}, \cdot)$ and a field X_K associated to Kraus operators, which in general cannot be described in terms of the bivectors Λ, R. Subject to this evolution, the density matrix will describe a curve $rho(t)$ over the manifold $D(\mathcal{H})$ of density matrices. This is an example of *stratified* manifold: it can be divided in strata according

[5] Note that in order not to use too much notation we are constantly making use of the isomorphism φ between $i\mathfrak{u}$ and $(i\mathfrak{u})^*$, so that we identify the density matrix with its corresponding element $\rho \in (i\mathfrak{u})^*$. Being explicit we would then have $K = \phi \circ \mathrm{ad}_H \circ \phi^{-1}$.

to the rank of the density matrix, from those of rank 1 (pure states $|\psi\rangle\langle\psi|$) to those of rank N remember the example of the Bloch sphere in chapter 1). The characteristics of the evolution associated to each field are different:

- A Hamiltonian field preserves the trace and the purity of a density matrix. We have seen that in the Bloch sphere purity corresponds to radius, hence such a field is tangential to constant radius surfaces.
- A gradient field does not preserve the trace of the matrix, but it preserves its rank (it is tangent to the strata). We cannot represent it easily on the Bloch sphere, since due to it not being trace-preserving, it points away of the three-dimensional hyperplane where the sphere is.
- The field associated to Kraus operators does not preserve the trace nor the rank of the matrix.
- The total field (sum of all three) does preserve the trace, since the effects of the gradient and Kraus fields cancel out (note that J depends on K_i in the right way for this to hold).However, the presence of the Kraus field causes that in general the rank of the matrix is not preserved and the states lose purity (decoherence): the field now points towards the inside of the Bloch sphere.

Once we have seen this, we know wonder:can we transport the time evolution that affects the density matrices to the space of tensors? Our idea is now to fix the density matrix and build a family of tensors $\{\Lambda^{(t)}\}_{t\geq 0}, \{R^{(t)}\}_{t\geq 0}$ which represents the time evolution of the system. Remember that these tensors encode the structure of the LJB algebra, our idea is to let this structure evolve:

$$[,] \to [,]_t \text{ tal que } [A(t), B(t)] = [A(0), B(0)]_t$$

Chruściński et al. apply this ideas to the associative product in [6]. This allows to study in other terms, for example, the purity of a state $\operatorname{tr}\rho^2(t) = \operatorname{tr}(\rho(t)\cdot\rho(t)) = \operatorname{tr}(\rho(0)\cdot_t \rho(0))$ with $\cdot_t$ a time dependent associative product.

This structure evolution has a natural interpretation in geometric terms. Consider a curve $\rho(t)$ in $(i\mathfrak{u})^*$. If this curve corresponds to the dynamics governed by a field X^L, i.e. if $\rho(t)$ is an integral curve of this field:

$$\frac{\partial\rho(t)}{\partial t} = X^L_{\rho(t)}$$

and ϕ_t is the associated flux ($\rho(t) = \phi_t(\rho_0)$), we can use it to transport the tensor along the integral curves of the field X^L, and this way obtain a time evolution

$$\Lambda^{(t)}_p(\alpha,\beta) = \Lambda^{(0)}_{\phi_t^{-1}(p)}(\phi_t^*(\alpha), \phi_t^*(\beta)) \tag{2.2}$$

for α, β to arbitrary forms and p any point of the manifold. Another way of seeing who the evolution of structures and tensors are related is:

$$\Lambda^{(0)}_{\rho(t)}(d\hat{A}, d\hat{B}) = (\rho(t), [A,B]) \to (\rho(0), [A,B]_t) = \Lambda^{(t)}_{\rho(0)}(d\hat{A}, d\hat{B})$$

From (2.2) it follows the differential equation which governs the evolution of tensors:

$$\frac{d}{dt}\Lambda^{(t)} = -\mathcal{L}_{X^L}\Lambda^{(t)} \qquad \frac{d}{dt}R^{(t)} = -\mathcal{L}_{X^L}R^{(t)}$$

where $\mathcal{L}_{X^L}$ is the Lie derivative operator with respect to the field X^L. The solution will be

$$\Lambda^{(t)} = e^{-t\mathcal{L}_{X^L}}\Lambda^{(0)} \qquad R^{(t)} = e^{-t\mathcal{L}_{X^L}}R^{(0)}$$

Since the flux ϕ_t is a diffeomorphism for any finite time t, the result of the evolution seems not to be too interesting: $\Lambda^{(t)}$ y $R^{(t)}$ are all diffeomorphic and the associated algebras are also pairwise isomorphic. That is why we will be much more interested by the behaviour at long times, which we will effectively represent as the limit $t \to \infty$. If it exists, we can define the limit structures

$$\Lambda^{\infty} = \lim_{t\to\infty}\Lambda^{(t)} \qquad R^{\infty} = \lim_{t\to\infty}R^{(t)}$$

which inform us about the system after a long time has passed. In particular, if the antisymmetric tensor loses components, the associated Lie algebra is more abelian, there are more operators that conmute, and the quantum character is supressed while a more classical behaviour emerges. It is worth to remark what happens in $i\mathfrak{u}$ when we make the tensors evolve in $(i\mathfrak{u})^*$. This is equivalent to having algebraic structures which depend on t. When we take the limit $t \to \infty$, in some cases the resulting algebra (associated to $\Lambda^{\infty}, R^{\infty}$) is different from the original one. This is known as *contraction of algebras*. In 1953 Inönü and Wigner used such a contraction to obtain the algebra of the Galilean transformation group from that of the Poincaré group by taking the limit of infinite speed of light c, $c \to \infty$ [12].
In the next chapter we will see some examples of what we proposed in this chapter.

Chapter 3

Examples

In this chapter we will put to use what we have seen in the previous ones, and we will try to find out when the proposed construction actually works in the $t \to \infty$ limit.

3.1 Decoherence in three levels

We choose to work a three-level system $\mathcal{H} = \text{span}\{|1\rangle, |2\rangle, |3\rangle\}$. The space of Hermitian operators has then dimension 9. To work in it we choose the basis of Gell-Mann matrices:

$$\{\lambda_i\} = \left\{ \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix} \right.$$
$$\left. \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \sqrt{\frac{1}{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}, \sqrt{\frac{2}{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right\}$$

By choosing this basis, we reduce the preservation of the trace of $\rho = \sum x_i \lambda_i$ to the conservation of its component x_9, since every matrix in the basis is traceless safe for λ_9, which is proportional to the identity. This, as we will see later, will decrease a little bit the complexity of the problem. Let us first focus on the antisymmetric tensor. Remember that Λ has the form:

$$\Lambda = \sum c_{ij}^k x_k \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j}$$

where c_{ij}^k are the structure constants which define the Lie algebra, and additionally they are the coordinates of Λ in the basis of linear antisymmetric bivectors $\{x_k \partial_i \wedge \partial_j\}$ (where we use $\partial_i = \frac{\partial}{\partial x_i}$ as an abbreviation). The evolution of Λ will take place in this linear space, hence it is convenient for us to ask ourselves what is dimension is. The answer is quite a non-negligible number: since we can assume $i < j$ ($\partial_i \wedge \partial_j = -\partial_j \wedge \partial_i$), we have $\dfrac{9(9-1)}{2} = 36$ possible values for the pair (i, j) and 9 values for k: the space of antisymmetric linear bivectors over $(i\mathfrak{u})^*$ has dimension $36 \cdot 9 = 324$. This forces us to use a symbolic calculus software to study this system. We use *Mathematica 9* to perform the calculations in what follows. Even in this case, every simplification of the problem is welcome, hence we note that it will always be $c_{i9}^k = 0$ since in our case λ_9 is proportional to the identity, a matrix that conmutes with any other matrix. By not considering the corresponding basis elements, we may restrict ourselves to a subspace $\mathcal{S} = \text{span}\{x^k \partial_i \wedge \partial_j\}_{1\leq i<j\leq 8,\, 1\leq k\leq 9}$ of dimension "only" 252.

Thanks to the symbolic calculus software we can compute the matrix of the operator $\mathcal{L}_{X^L}$ acting on this subspace. Remember that the physical information of our system is contained in X^L, the Lindblad field, which is linear and has the appropiate form (seen in section 2.3, from the results by Gorini, Kossakowski and Sudarshan) so that the operator can be restricted to the subspace we are working in

$$\mathcal{L}_{X^L} : \mathcal{S} \longmapsto \mathcal{S}$$

The limit Λ_∞ of the tensor evolution will exist if it exists $\lim_{t\to\infty} e^{-t\mathcal{L}_{X^L}}$, for which it is a sufficient condition that the eigenvalues of the matrix representation of $\mathcal{L}_{X^L}$ over $\mathcal{S}$ all have positive real parts. For a matrix of dimension 252 this seems a lot to ask for, and indeed, we check that for many simple fields X^L the operator $\mathcal{L}_{X^L}$ has eigenvalues with both positive and negative real parts.

The solution to this problem comes from repeating the former strategy: we must restrict ourselves to a smaller subspace. Hence the necessary and sufficient condition for the limit to exist is the existence of a subspace $\mathcal{S}_+ \subset \mathcal{S}$ such that (i) it contains the initial condition of our dynamics ($\Lambda \in \mathcal{S}_+$), (ii) it is invariant under the operator $\mathcal{L}_{X^L}$ ($\mathcal{L}_{X^L}(\mathcal{S}_+) \subset \mathcal{S}_+$) and (iii) the eigenvalues of the restriction of $\mathcal{L}_{X^L}$ to S_+ have positive real parts.

We start hence with an example taken from [6], in which a particle with a finite and discrete spectrum suffers decoherence. We take:

$$L\rho = -\gamma[X,[X,\rho]] \qquad \text{con } \gamma > 0,\ X = \sum_{m=1}^{3} m|m\rangle\langle m|$$

The interested reader can check that indeed this field is of the form (2.1) with only one Kraus operator $K = \sqrt{2\gamma}X$. If we have it act on $|m\rangle\langle n|$ we get

$$L|m\rangle\langle n| = -\gamma(m-n)^2|m\rangle\langle n|$$

Therefore it is patent that the particle is suffering decoherence, since every off-diagonal term of the density matrix will decay exponentially. This can also be clearly seen in the corresponding field X^L which we write:

$$X^L = -\gamma\partial_1 - \gamma\partial_2 - 4\gamma\partial_4 - 4\gamma\partial_5 - \gamma\partial_6 - \gamma\partial_7$$

Indeed,the components associated to the non-diagonal elements of the basis (all but λ_3, λ_8 and λ_9) go to zero, and ρ is ieft asymptotically diagonal. Once the dynamics of ρ is understood, let us see if we can transfer it to the tensors. If we compute the time dependent structure constants $c_{ij}^k(t)$ in this example, we find a problem, or, to be precise, four problems:

$$c_{1,6}^5(t) \to \infty \qquad c_{1,7}^4(t) \to -\infty \qquad c_{2,6}^4(t) \to -\infty \qquad c_{2,7}^5(t) \to -\infty$$

If we diagonalize the matrix of the operator we find the cause. In $\mathcal{S}$ there exists an invariant subspace $\mathcal{S}_-$ of dimension 34 associated to eigenvalues with negative real part. The initial tensor

Λ turns out to have nonvanishing projection on this subspace given by

$$\frac{x_5}{2}\partial_1 \wedge \partial_6 - \frac{x_4}{2}\partial_1 \wedge \partial_7 - \frac{x_4}{2}\partial_2 \wedge \partial_6 - \frac{x_5}{2}\partial_2 \wedge \partial_7$$

Consequently, there exists no invariant subspace $\mathcal{S}_+$ with eigenvalues that have positive real parts that contains Λ. Note that this does not mean that the evolution that this field induces in the density matrices is not convergent, only that if we choose to try and transfer this dynamics to the tensors, their long time limit, at least for the antisymmetric tensor, is not well defined. The treatment of this example done in [4] is somewhat different, since the operator used is:

$$L|m\rangle\langle n| = -4\gamma \sin^2\left(\frac{(m-n)\pi}{3}\right)|m\rangle\langle n|$$

(The main difference is that we now consider $\{|m\rangle\}$ as the discretized position eigenstates of a particle on a circle instead of a line, what introduces an extra symmetry in the system). This way, the field is

$$X^L = -3\gamma\partial_1 - 3\gamma\partial_2 - 3\gamma\partial_4 - 3\gamma\partial_5 - 3\gamma\partial_6 - 3\gamma\partial_7$$

and now Λ converges. An analogous treatment reveals that R also converges. In addition, the Lie algebra associated to Λ^∞ and the Jordan algebra associated to R^∞ are compatible in a way that they define a LJB algebra over the limit submanifold.

3.2 General case for real diagonal Kraus operators

Motivated by the former examples and given that it is a simple case inside the vast amount of possible dynamics for a physical system, we are going to try to totally understand the convergence of tensors under a field given by an arbitrary number of real diagonal Kraus operators in three dimensions. Let then

$$K_i = \sqrt{2}\begin{pmatrix} a_i & 0 & 0 \\ 0 & b_i & 0 \\ 0 & 0 & c_i \end{pmatrix} \qquad a_i, b_i, c_i \in \mathbb{R}, i = 1, \ldots, n$$

where we have included a factor $\sqrt{2}$ for convenience for the computations that follow and let us consider the Lindblad operator:

$$L\rho = \sum_i K_i \rho K_i^\dagger + J \circ \rho$$

with $J = \sum_i K_i^\dagger K_i$ the one corresponding to our choice of Kraus operators. We start with the simplest case, $n = 1$. The Lindblad field is

$$X^L = -\gamma_a\partial_1 - \gamma_a\partial_2 - \gamma_b\partial_4 - \gamma_b\partial_5 - \gamma_c\partial_6 - \gamma_c\partial_7 \tag{3.1}$$

where

$$\gamma_a = (b_1 - a_1)^2 \qquad \gamma_b = (a_1 - c_1)^2 \qquad \gamma_c = (c_1 - b_1)^2$$

From here we can deduce the form of the Lindblad field for any n. Indeed, if $L = \sum_i L_i$ with $L_i\rho = K_i\rho K_i^\dagger + (K_i^\dagger K_i)\circ\rho$, it is not difficult to check the following two additivity properties:

$$X^L = \sum_i X^{L_i} \qquad \mathcal{L}_{X^L} = \sum_i \mathcal{L}_{X^{L_i}}$$

Hence in general the field will have the form of (3.1) with

$$\gamma_a = \sum_i (b_i - a_i)^2 \qquad \gamma_b = \sum_i (a_i - c_i)^2 \qquad \gamma_c = \sum_i (c_i - b_i)^2 \tag{3.2}$$

Let us consider the case of the antisymmetric tensor. With this field, the eigenvectors of $\mathcal{L}_{X^L}$ turn out to be precisely the bivectors of our basis, hence we have the great advantage that the matrix of the operator is diagonal. Indeed, this is satisfied for any field of the form:

$$X = -\sum_l \gamma_l x_l \partial_l$$

We can prove it and at the same time compute the associated eigenvalue:

$$\begin{aligned}\mathcal{L}_X(x_k\partial_i \wedge \partial_j) &= \sum_l \mathcal{L}_{-\gamma_l x_l \partial_l}(x_k\partial_i \wedge \partial_j) = \sum_l -\gamma_l\left([x_l\partial_l, x_k\partial_i] \wedge \partial_j + x_i\partial_i \wedge [x_l\partial_l, \partial_j]\right) = \\ &= (-\gamma_k + \gamma_i + \gamma_j)x_k\partial_i \wedge \partial_j\end{aligned}$$

The following table displays all the elements of the basis on which Λ has nonvanishing projection, and the associated eigenvalue, asumiendo ya $\gamma_1 = \gamma_2 = \gamma_a$,$\gamma_4 = \gamma_5 = \gamma_b$ y $\gamma_6 = \gamma_7 = \gamma_c$:

i,j,k	Eigenvalue	i,j,k	Eigenvalue	i,j,k	Eigenvalue
1,2,3	$2\gamma_a$	2,6,4	$\gamma_a + \gamma_b - \gamma_c$	4,7,1	$-\gamma_a + \gamma_b + \gamma_c$
1,3,2	0	2,7,5	$\gamma_a - \gamma_b + \gamma_c$	4,8,5	0
1,4,7	$\gamma_a + \gamma_b - \gamma_c$	3,4,5	0	5,6,1	$-\gamma_a + \gamma_b + \gamma_c$
1,5,6	$\gamma_a + \gamma_b - \gamma_c$	3,5,4	0	5,7,2	$-\gamma_a + \gamma_b + \gamma_c$
1,6,5	$\gamma_a - \gamma_b + \gamma_c$	3,6,7	0	5,8,4	0
1,7,4	$\gamma_a - \gamma_b + \gamma_c$	3,7,6	0	6,7,3	$2\gamma_c$
2,3,1	0	4,5,3	$2\gamma_b$	6,7,8	$2\gamma_c$
2,4,6	$\gamma_a + \gamma_b - \gamma_c$	4,5,8	$2\gamma_b$	6,8,7	0
2,5,7	$\gamma_a + \gamma_b - \gamma_c$	4,6,2	$-\gamma_a + \gamma_b + \gamma_c$	7,8,6	0

If we want the antisymmetric tensor to converge, all of the listed eigenvalues must be bigger or equal to zero. We must hence demand the following set of inequalities:

$$0 \le \gamma_a \le \gamma_b + \gamma_c \qquad 0 \le \gamma_b \le \gamma_c + \gamma_a \qquad 0 \le \gamma_c \le \gamma_a + \gamma_b \tag{3.3}$$

Equivalently $\gamma_a, \gamma_b, \gamma_c$ should be the sides of a triangle. To see what this conditions means in terms of Kraus operators, consider the (affine) points $A \equiv \vec{a} = (a_1, \dots, a_n), B \equiv \vec{b} = (b_1, \dots, b_n), C \equiv \vec{c} = (c_1, \dots, c_n) \in \mathbb{R}^n$. The sides of the triangle they define are, due to (3.2):

$$\|\vec{b} - \vec{a}\| = \sqrt{\gamma_a} \qquad \|\vec{a} - \vec{c}\| = \sqrt{\gamma_b} \qquad \|\vec{c} - \vec{b}\| = \sqrt{\gamma_c}$$

Hence the conditions (3.3) turn into

$$\|\vec{b}-\vec{a}\|^2 \leq \|\vec{a}-\vec{c}\|^2+\|\vec{c}-\vec{b}\|^2 \qquad \|\vec{a}-\vec{c}\|^2 \leq \|\vec{c}-\vec{b}\|^2+\|\vec{b}-\vec{a}\|^2 \qquad \|\vec{c}-\vec{b}\|^2 \leq \|\vec{b}-\vec{a}\|^2+\|\vec{a}-\vec{c}\|^2$$

But because of the cosine law this becomes

$$\cos\widehat{ACB} \geq 0 \qquad \cos\widehat{CBA} \geq 0 \qquad \cos\widehat{BAC} \geq 0$$

Hence we reach the following somewhat more concise that intuitive conclusion:

> *The antisymmetric tensor converges for evolution given by Kraus operators K_i if and only if the points A, B, C define an acute triangle.*

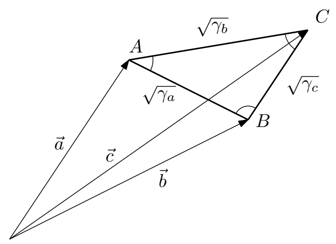

Figure 3.1: Graphical representation employed to enunciate the convergence conditions.

By a similar procedure we check that the convergence condition is the same for the symmetric tensor R associated to the Jordan structure.
There is a couple of consequences we can draw from this result. The first one corresponds to the case $n = 1$, only one Kraus operator. The cited triangle is then obviously degenerate, since it is contained in $\mathbb{R}$, and the convergence condition is reduced to at least two of the points A, B, C being the same, i.e., at least one γ is 0 (if the three points coincide the Kraus operator is a multiple of the identity, and the operator L vanishes). On the other hand, the condition and characteristics of convergence (which only depend on $\gamma_a, \gamma_b, \gamma_c$) are invariant by translation and rotation of the triangle, hence we can always consider one of the vertices to be the origin, another one to be on the x-axis, and the third one on the xy-plane, so that only a maximum of two Kraus operators are needed to get a given field (as long as it is of the form (3.1) and $\sqrt{\gamma_a}, \sqrt{\gamma_b}, \sqrt{\gamma_c}$ can be the sides of a triangle):

$$K_1 = \sqrt{2}\begin{pmatrix} a_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, K_2 = \sqrt{2}\begin{pmatrix} a_2 & 0 & 0 \\ 0 & b_2 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \begin{aligned} a_1 &= \frac{\gamma_c+\gamma_b-\gamma_a}{2\sqrt{\gamma_c}} \\ a_2 &= \sqrt{\gamma_b - a_1^2} \\ b_2 &= \sqrt{\gamma_c} \end{aligned}$$

In this manner, for the example taken from [4] this operators are enough:

$$K_1 = \sqrt{2}\begin{pmatrix} \frac{\sqrt{3\gamma}}{2} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, K_2 = \sqrt{2}\begin{pmatrix} \frac{3\sqrt{\gamma}}{2} & 0 & 0 \\ 0 & \sqrt{3\gamma} & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

and we have convergence since the triangle is equilateral.

Conclusiones

What have we done?

In this work we have presented a technique to study time evolution in open systems, which consists in transferring the dynamics from one set of mathematical objects (density matrices) to another one (algebraic structures defined on them). We have cared specially for the existence of the long time limit of such dynamics, with the hope of extracting conclusions in the cases in which such limit is nontrivial. To this respect, we have checked that convergence of the tensors is not guaranteed, even though the evolution in the space of density matrices is perfectly well-defined and convergent. We have seen as well that in certain particular cases we can characterize the systems whose tensors converge.

What is to be done yet?

The study of tensor dynamics promises to be broad and offer very diverse and interesting situations to be analysed. The last convergence characterization presented in chapter 3 is susceptible of being generalized by considering an arbitrary number of levels in the system, or relaxing the conditions on the Kraus operators. For example, if we consider complex operators, the vector field is no longer as simple and the bivector basis we have been using is no longer an eigenbasis, except in very particular cases. It would also be very interesting to try to understand the physical meaning of the convergence characterization that we have found (and which basically seems to mean that there cannot be a decay rate that is much quicker than others), as well as the consequences, if any, that the existence of the limit has for the behaviour of the system. More in general, the analysis of the limit structures Λ^∞, R^∞, and the information that can be obtained from them should be performed. As an example, given the limit Lie algebra we should be able to compute, from its Casimir operators (those which conmute with every element in the algebra), conserved quantities of the dynamics on the limit submanifold.

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