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Chasing a cMERA

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Chasing a cMERA

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The continuous Multiscale Entanglement Renormalization Ansatz provides a variational ansatz for the ground state of a quantum field theory. Not only that, it provides a UV regularization scheme, since cMERA states come equipped with an intrinsic UV cutoff. We review this construction for one-dimensional free CFTs, and characterize the correlations and the entanglement entropy profile of the cMERA corresponding to the ground state of a 1+1 Dirac free fermion theory. Our results reveal how the UV regularization is reflected in the entanglement structure of the cMERA, and leave the door open for further applications.

1 Introduction

Not in few occasions progress in physics is achieved when techniques and insights from a particular area are applied to tackle problems in another one which at first might seem independent. The present essay addresses recent advances in the useful interplay between quantum field theory, on one side, and quantum information and quantum many-body theory on the other, in the framework of continuous tensor networks.

Quantum field theory is one of the most succesful human attempts to describe nature at a microscopic level. It is a common tool in fields ranging from fundamental particle theory to condensed matter physics. It presents, nevertheless, several limitations that often lead to ill-defined concepts and cause confusion. One of the issues present in many quantum field theories is the need for **UV regularization** of the theory[1], i.e. the inclusion of some form of cutoff that prevents modes of arbitrarily high energy from affecting the predictions for observables of the theory, hence avoiding likely divergences in these. These modes are considered to be outside the scope of our theory, in the philosophy of effective field theories and renormalization. One of the simplest methods to carry out this regularization is the discretization of the

[1] Most commonly known among students as "sweeping the theory's infinities under the rug".

spatial support of the theory, that is to say, placing the theory in a lattice of finite lattice spacing a. This lattice spacing provides a UV cutoff that prevents high energy modes from kicking in. For example, in a free theory like the ones we will consider in the main part of this essay, high energy modes directly correspond to high momentum modes, but momenta higher that $\frac{2\pi}{a}$ are forbidden in our lattice approach. Even though effective, placing a quantum field theory on the lattice turns out to be somewhat drastic in other aspects. Another common feature of the theories we are going to consider is that they are **conformal field theories** (CFTs). These are quantum field theories whose symmetries include the full conformal group of the space they are defined in, and they are particularly useful to represent critical systems at continuous phase transitions, due to their conformal symmetry removing any dependence on a physical scale. Placing a CFT on a lattice suddenly removes the rich spectrum of symmetries of the theory, reducing, for instance, full translational and rotational symmetries to scarce discrete subgroups of them.

Quantum **entanglement**, on the other hand, is a concept that seems to be playing a very fundamental role in current and possibly future theoretical physics. It is usually described, somewhat imprecisely, as the property of quantum mechanics that full knowledge of the state of a system does not imply full knowledge of the state of its subsystems. Entanglement is a many-faced phenomenon that has been devoted attention as a possible guideline to distinguish classical from quantum, in quantum foundations; as the source of emergent spacetime, in quantum gravity; and less speculatively, as a useful tool for the characterization of many body states in quantum many-body theory and condensed matter theory, and as a resource to improve our information treatment properties, in quantum information theory. For us, entanglement will constitute an important characteristic of the quantum states we will be dealing with, and we will appreciate it in the context of **entanglement renormalization**.

A useful concept to study these areas of physics is that of **scale transformations** and **renormalization group flows**. Indeed, scale transformations play an important role in the Wilsonian renormalization scheme for quantum field theories, and also in Kadanoff's block spin renormalization approach to many body spin systems on a lattice. Even though a rigorous definition of a scale transformation is something that still has not been agreed upon, everyone has an intuitive idea of what scaling a system is, even if only based on the zooming and finite resolution properties of optical devices, our eyes among them. Standard renormalization procedures on a lattice involve its coarse-graining into an effective smaller set of degrees of freedom that represent the system after *zooming out*. The concept of entanglement renormalization is based on the idea that naïve coarse-graining schemes fail to address the entanglement present at the length scales that are removed in the scale transformations, leading to its accumulation in the coarse-grained versions of the system and thus to an improper RG flow. Its proposal is that a consistent implementation of RG scale transformations must deal with the entanglement properties of the system, making sure that short-range entanglement is removed upon coarse-graining. Conversely, we can flow in the opposite direction and reintroduce entanglement at shorter and shorter length scales. This very simple idea led to a series of useful insights, including the Multiscale Entanglement Renormalization Ansatz (MERA), which has been the object of intensive research over the last ten years. Much less well-known is its continuous, field theoretical counterpart (cMERA), which is the object of study of this essay.

Guide for the reader

Now we have all the pieces of the puzzle for us to fit them together and give the starting point for the essay. We will begin by reviewing the MERA, what it represents and what it is useful for. Next, we will talk about cMERA and its relation to MERA, and we will review the cMERA structures for one dimensional bosonic and fermionic CFTs. Once this background has been introduced, we will present our results: we will analyze the correlations in a cMERA state for a 1+1 Dirac free fermion theory and we will study and give expressions for the scaling of entanglement entropy in such a state. This provides a check for the cMERA formalism in a well-understood environment like that of a free CFT. Success in this goal may however lead to important advances in obtaining a sound and satisfactorily well-defined continuous tensor network approach to non-interacting and interacting quantum field theories alike. But every long journey begins with a first step, so let's begin!

2 MERA and cMERA

As we have mentioned in the introduction, cMERA arises as a generalization to the continuum of the MERA tensor network, so let us first see how this is done and why we should care about it.

2.1 MERA

Tensor networks were introduced in the context of many-body quantum theory to provide us with computationally tractable approximations to ground states of local hamiltonians. In general, the state of a system with N degrees of freedom, for example qudits (d-level quantum systems), will be determined by the exponentially growing amount of d^N complex coefficients[2], which we arrange in a multi-indexed array called a tensor. Tensor networks are particular ways of decomposing this

[2] One less than that if we want to be extremely precise and take into account the normalization and global phase freedom.

huge tensor into contractions of smaller tensors such that the number of parameters involved in the description is considerably reduced. This is usually represented graphically in diagrams like the one of figure 1, where the indices of a tensor are drawn as "legs" coming out of its body, which can be joined (contracted) with those of other tensors. From this point of view, a tensor network is useful if it allows for an efficient representation of such ground states and it mimics their properties, e.g., the behaviour of correlations and entanglement in the system, allowing as well for an efficient determination of expectation values of local insertions of observables of the theory.

But the interest in tensor networks does not solely arise from its computational advantages as a variational class. They have proved themselves useful in analyzing the entanglement structure of quantum states, as well as in the study of quantum phase transitions and more recently, in the particular case of MERA, in the holographic approach to quantum gravity given by the AdS/CFT correspondence, as we mention below.

To begin with, a MERA that prepares a state on a 1-dimensional lattice is a 2-dimensional tensor network as the one shown in figure 1 [1, 2]. Its horizontal dimension represents the spatial dimension of the system, while its vertical dimension is associated to scale. Generalizations of the same scheme to higher dimensional systems are straightforward. As a variational class it is particularly appropiate to describe critical phenomena, for it displays power-law decaying correlations and logarithmic scaling of entanglement entropies. Indeed, it has been shown that MERA allows for the extraction of conformal data of an underlying CFT directly from the lattice. When all the tensors in the MERA are equal regardless of which level the are in, we say the MERA is *scale invariant*.

MERA can be interpreted in different ways, but the one that will be most relevant for us consists of regarding it as a quantum circuit which, starting from a product state, generates the final approximation to the ground state via a series of two-qubit unitary gates (figure 2). In order to better understand this picture, let us first learn about the integrants of MERA. A MERA is composed of two kinds of tensors: unitaries (usually called *disentanglers*) and *isometries*. The disentanglers implement a unitary transformation between the Hilbert space associated to their two upper legs and the one associated to the lower ones. The isometries implement, of course, an isometric map between the Hilbert space associated to their only upper leg and the necessarily bigger space associated to the lower ones. Start from the top of the MERA, and consider the following two step process: from one layer to the next, the action of the isometries introduces new degrees of freedom in the system, interspersed with the pre-existing ones. In fact, we can see the isometries as two-site to two-site unitaries, like the disentanglers, one of whose inputs was originally in some ancillary product state $|0\rangle$ with the rest of the system. After

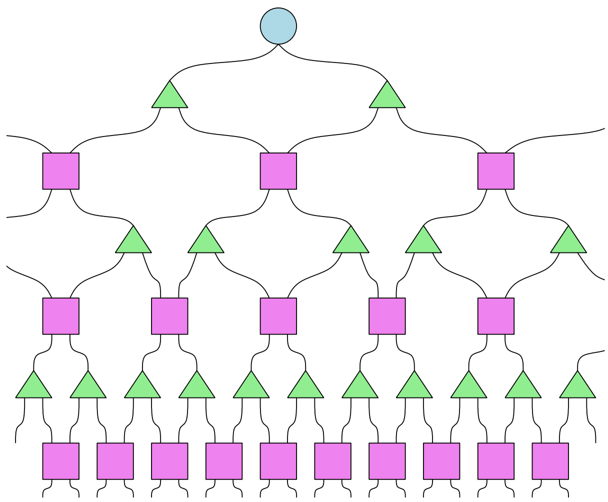

Figure 1: A MERA tensor network. Isometries are shown as green triangles, and disentanglers as violet squares.

the isometries, the disentanglers[3] entangle the newly introduced degrees of freedom with the previously existing ones. This generation of short range entanglement can be interpreted as resulting from the action, during a short period of time, of some local interaction Hamiltonian. After these two steps we find ourselves in a situation similar to the one at the beginning of step one, and we can proceed in the same fashion. The result: we obtain our final state through local unitary operations on an initial product state that introduce entanglement at different scales, depending on their vertical coordinate in the network.

We can give an alternative interpretation of MERA via a block coarse-graining procedure. In this case we start from the bottommost layer, the state on the lattice, and divide it into two site blocks. Now we begin to move upwards for the RG flow. The application of the first layer of disentanglers removes the short range entanglement between neighbouring sites of the lattice that belong to different blocks. This is of the utmost importance for the next step, where isometries map states in a two-site Hilbert space to a one-site Hilbert space in the layer above, effectively removing part of the information of the previous layer and thus realizing the coarse-graining operation. This can be understood as a rescaling transformation in which correlations

[3]In this picture we should call them indeed entanglers. The reason for this now confusing nomenclature stems from the entanglement renormalization picture of the MERA, which we will address soon.

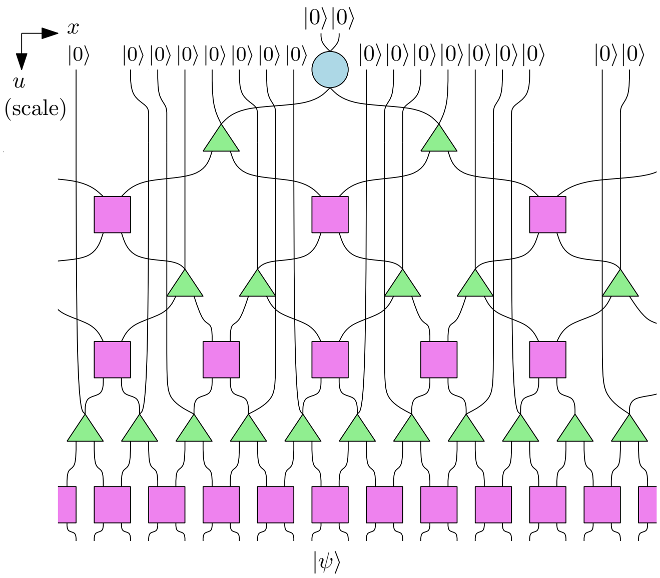

Figure 2: MERA as a quantum circuit.

at shorter range than a certain scale are removed, hence the name *entanglement renormalization*.

Hence MERA provides a computationally tractable ansatz for the ground state of a (lattice regularized) CFT; it brings insight about its entanglement structure, via its quantum circuit picture, and it implements an entanglement renormalization RG flow on the lattice. These are enough reasons to be willing to find a continuous version of it even without counting that, excitingly enough, it also realizes the holographic principle. The extra dimension in AdS/CFT and the vertical dimension in MERA both have a meaning of scale. In fact, MERA has been proposed, first, as a discretization of AdS, and later of its space of geodesics, to which it maps under an X-ray transform. This is currently a very active area of research [3, 4, 5, 6].

2.2 cMERA

The *continuous MERA* or *cMERA* provides a representation for the ground state of a quantum field theory inspired in the philosophy of MERA in the lattice. This construction was introduced in [7], and its analogy with the discrete case can be better understood via the quantum circuit picture of MERA described in the previous section, where we started from a product state and applied a series of unitaries which progressively introduce entanglement at different scales. cMERA generalizes this same idea by making the two dimensions of the MERA continuous:

- First, we replace the discrete entangling steps by a unitary dependent of a continuous scale parameter $u \in \mathbb{R}$ (continuous version of the *vertical* dimension of the MERA).
- Second, we replace our finite lattice quantum system by a quantum field theory (continuous version of the *horizontal* dimension of the MERA). As it happened on the lattice, we expect our cMERA to represent the ground state of a CFT, which will in turn represent a critical state of a continuous phase transition.

Note that, these two modifications being independent, we can afford to take one and not the other. In particular, considering a continuous scale parameter in a discrete lattice system provides us with the *lattice cMERA*, which is itself an interesting construction. We are not going to deal with it directly, but it will appear naturally in practical numerical implementations of the cMERA.

Just as its discrete counterpart, cMERA will start from a product state $|\Omega\rangle$, with no entanglement between its constituents, and will generate a new state through an entangling unitary evolution in scale $U(u_1, u_2)$ defined as

$$U(u_1, u_2) = \mathcal{P} \exp\left(-i \int_{u_1}^{u_2} du\, (K(u) + L)\right) \qquad u_2 \geq u_1 \tag{1}$$

where $\mathcal{P}$ denotes the scale path ordering of the integral. The generator of this evolution is composed itself of two operators: L and $K(u)$. Both are local functions of the fields of the theory. The first one, L is the responsible of the rescaling of space, and of the operators. It generates transformations of the form:

$$x \longmapsto \lambda x \qquad \mathcal{O} \longmapsto \lambda^d \mathcal{O} \tag{2}$$

The second part of the scale transformation generator is $K(u)$, known as the **entangler**. Its role in the cMERA paradigm is analogous to the role of the (dis)entanglers of MERA: it introduces short range entanglement during the scale evolution, and so to say compensates for the effect of L, which increases the range of the preexisting correlations. Just as in a scale invariant MERA all the tensors are equal regardless of the level at which they are placed, for a conformally invariant (and hence scale independent) CFT, we will demand K to be independent of u. Note that this saves us the inclusion of the path ordering in the definition of U above. To build a cMERA state, we depart from the uncorrelated product state $|\Omega\rangle$ and start introducing entanglement from the IR scale limit $u = -\infty$ to some finite u_0. Note that since there exists reparametrization freedom of the scale parameter u, we can fix our convention by choosing $u_0 = 0$ and define our cMERA as

$$|\psi(u)\rangle = U(u, -\infty)\,|\Omega\rangle \implies |\psi\rangle_{\text{cMERA}} = |\psi(0)\rangle = U(0, -\infty)\,|\Omega\rangle \equiv |\psi\rangle \tag{3}$$

Note that there is, up to what we have seen, a slight difference between cMERA and

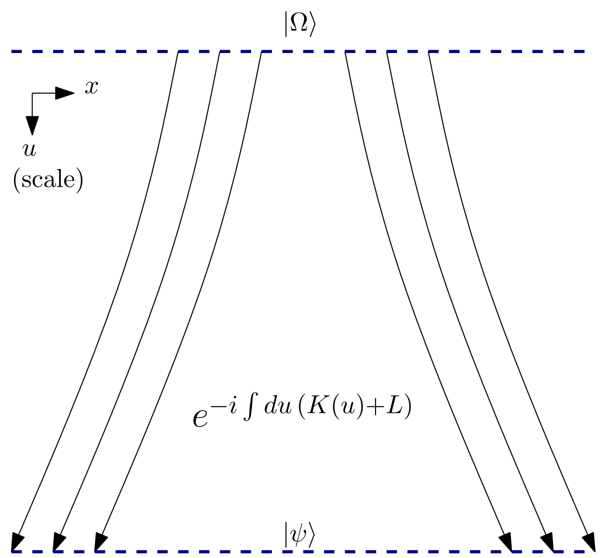

Figure 3: Scale evolution that originates the cMERA (to be compared with figure 2).

MERA, and that is that the mapping between the system at two distinct scales u_1, u_2 is a unitary transformation $U(u_2, u_1)$ while on the lattice it was a mere isometry. At this point it is a matter of current investigation whether this will suppose a problem in the applications of cMERA, and truncation schemes involving a smearing of the fields of the theory that return the isometric character to cMERA are being developed as these lines are written. For the purposes of this essay it is nevertheless safe to ignore this point and so will we do.

Stopping the evolution at a finite scale naturally introduces an energy scale cutoff in $|\psi\rangle$. Only when probing it at momenta smaller than the one corresponding to $u_0 = 0$, which we denote by Λ, the state we have built will look like the ground state of the target CFT. When examined at momenta larger than Λ (or equivalently distances shorter than $1/\Lambda$), we will hopefully find that no entanglement has been introduced at those scales, and the state still looks similar to a product state. Only if we continue the scale evolution until $u = \infty$, we would have introduced entanglement at all scales, and the result should be the ground state $|0\rangle$ of the target Hamiltonian, the one that characterizes our theory of interest. Thus cMERA manages to "interpolate" between the state $|0\rangle$ at long length scales and the state $|\Omega\rangle$ at short length scales[4]. This apparently simple statement contains the true essence of what the cMERA formalism accomplishes: namely a UV-regularization procedure for quantum field theories which preserves their continuous character more successfully than discretization to a lattice. As a matter of fact, cMERA has been seen to realize full conformal invariance on the cMERA [8], but it might require, nevertheless, the redefinition of what we mean, by, for example, a scale transformation. As we will see, the cMERA state will turn out to be invariant under scale transformations, provided that we take them to be generated by $K + L$ (rather than L, which would be the generator for the "naïve" scale transformation):

$$|\psi(u)\rangle = e^{-iu(K+L)}\,|\psi(0)\rangle = |\psi(0)\rangle \tag{4}$$

Let's now see how we build the cMERA formalism ($|0\rangle$, $|\Omega\rangle$, $|\psi\rangle$, L and K) in two particular, simple CFTs.

2.3 A first example: 1+1 free bosons

Let us begin with a concise reminder of what this massless, Klein-Gordon theory is about. Consider a single scalar field $\phi(x)$ in one dimension and its conjugate momentum $\pi(x)$, with whom it satisfies canonical commutation relations

$$[\phi(x), \pi(y)] = i\delta(x - y) \tag{5}$$

Let also the Fourier transformed operators be

$$\phi(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dx\, e^{-ikx}\phi(x)dx \qquad \pi(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dx\, e^{-ikx}\pi(x)dx \tag{6}$$

$$\phi(k)^\dagger = \phi(-k) \qquad \pi(k)^\dagger = \pi(-k) \qquad [\phi(k), \pi(q)] = i\delta(k + q) \tag{7}$$

Then we can express the Hamiltonian of the theory as follows:

$$H = \frac{1}{2}\int_{-\infty}^{\infty} dx(\pi(x)^2 + (\partial\phi(x))^2) = \int_{-\infty}^{\infty} dk(\pi(k)\pi(-k) + k^2\phi(k)\phi(-k)) \tag{8}$$

Now we can proceed to diagonalize this Hamiltonian in the usual way, by defining creation and annihilation operators $a(k), a^\dagger(k)$:

$$a(k) = \sqrt{\frac{|k|}{2}}\phi(k) + i\sqrt{\frac{1}{2|k|}}\pi(k) \qquad a^\dagger(k) = \sqrt{\frac{|k|}{2}}\phi(-k) - i\sqrt{\frac{1}{2|k|}}\pi(-k) \tag{9}$$

[4] On a less serious note here, the spelling and pronunciation of cMERA remind us of the Greek mythological monster Chimera ($\chi\iota\mu\alpha\iota\rho\alpha$), which amusingly enough also consisted of an "interpolation" of, in this case, a lion, a goat and a snake (hopefully more ferocious beasts to tame than our product and entangled states).

$$[a(k), a^\dagger(q)] = \delta(k+q) \tag{10}$$

in terms of which the Hamiltonian can be expressed as

$$H = \int_{-\infty}^{\infty} dk\, |k| a^\dagger(k) a(k) \tag{11}$$

The ground state of the system will be the one annihilated by all the annihilation operators:

$$a(k)\,|0\rangle = 0 \qquad \forall k \in \mathbb{R} \tag{12}$$

This state $|0\rangle$ contains, as is well known, entanglement at all length scales, and would be the UV limit of our entangling procedure. Now it remains to define the departure product state $|\Omega\rangle$. Since it is complicated to rigorously give a tensor product structure to the Hilbert space of a quantum field theory, which would be useful in order to define our product state with respect to it, we proceed from the finite dimensional case and take a continuous limit. Consider a discrete system which associates an independent, bosonic degree of freedom (harmonic oscillator) ϕ_n to every point in a lattice indexed by n. Since these degrees of freedom do not interact, the Hamiltonian will be a sum of self-energy terms:

$$H = \frac{1}{2}\sum_n \left(\pi_n^2 + \Lambda^2 \phi_n^2\right) \tag{13}$$

where Λ is some constant related to the frequency of the harmonic oscillator. The continuum limit of this system will be given by the following Hamiltonian:

$$H = \frac{1}{2}\int dx\, (\pi(x)^2 + \Lambda^2\phi(x)^2) = \frac{1}{2}\int dk\, (\pi(k)\pi(-k) + \Lambda^2\phi(k)\phi(-k)) \tag{14}$$

This Hamiltonian, just as its discrete counterpart, is already diagonal in real space, with creation and annihilation operators given by:

$$a(x) = \sqrt{\frac{\Lambda}{2}}\phi(x) + i\sqrt{\frac{1}{2\Lambda}}\pi(x) \qquad a^\dagger(x) = \sqrt{\frac{\Lambda}{2}}\phi(x) - i\sqrt{\frac{1}{2\Lambda}}\pi(x) \tag{15}$$

in such a way that the ground state of the theory is the common kernel of the annihilation operators. No surprise here, since this is nothing but the continuum limit of the tensor product of vacua of individual harmonic oscillators

$$|0\rangle = \bigotimes_n |0\rangle_n \qquad \text{where} \qquad \left(\sqrt{\frac{\Lambda}{2}}\phi_n + i\sqrt{\frac{1}{2\Lambda}}\pi_n\right)|0\rangle_n = 0 \tag{16}$$

which is the ground state of the lattice theory. To better compare with the ground state we found a page ago, though, let us express the creation and annihilation operators in momentum space

$$a(k) = \sqrt{\frac{\Lambda}{2}}\phi(k) + i\sqrt{\frac{1}{2\Lambda}}\pi(k) \qquad a^\dagger(k) = \sqrt{\frac{\Lambda}{2}}\phi(-k) - i\sqrt{\frac{1}{2\Lambda}}\pi(-k) \tag{17}$$

Compare this to equation (9). In both cases, the operators have the form

$$a(k) = \sqrt{\frac{\alpha(k)}{2}}\phi(k) + i\sqrt{\frac{1}{2\alpha(k)}}\pi(k) \qquad a^\dagger(k) = \sqrt{\frac{\alpha(k)}{2}}\phi(-k) - i\sqrt{\frac{1}{2\alpha(k)}}\pi(-k) \tag{18}$$

for a certain function $\alpha(k)$: a linear function $\alpha(k) = |k|$ for the CFT, a constant $\alpha(k) = \Lambda$ in the case of the product state. Remembering the properties we want of our cMERA, it seems natural to define it as the common kernel of the annihilation operators from (18), for some $\alpha(k)$ which interpolates between $|k|$ at small values of k and a constant at large values of k, where the terms "small" and "large" are meaningful with respect to a cutoff Λ. The simplest choice, which we will call the *sharp regularization scheme* is then:

$$\alpha(k) = \begin{cases} |k| & |k| \leq \Lambda \\ \Lambda & |k| > \Lambda \end{cases} \tag{19}$$

which is represented in figure 4. The cMERA hence turns out to be the ground state of the following Hamiltonian:

$$H = \int_{-\infty}^{\infty} dk\, k \left(\frac{1}{\alpha(k)}\pi(k)\pi(-k) + \alpha(k)\phi(k)\phi(-k) \right) \tag{20}$$

This characterization of cMERA is different than the one we gave in the previous section. Both can be shown to be equivalent by dualizing the entangling scale evolution to the operators, and seeing that for the appropriate choice of K, the annihilation operators of $|\Omega\rangle$ (equation 17) evolve into those of cMERA (equation 18). Let us then see what are L and K in this theory. The generator of standard scaling transformations is given by:

$$L = -\frac{1}{2}\int_{-\infty}^{\infty} dx\, \left(\pi(x)x\partial\phi(x) + x\partial\phi(x)\pi(x) + q(\psi(x)\pi(x) + \pi(x)\phi(x))\right) \tag{21}$$

where q is a parameter that will determine the scaling dimensions of ϕ and π. Indeed, we can compute the result of infinitesimally evolving these fields:

$$\frac{\partial\phi(x,u)}{\partial u} = -i[L_q, \phi(x)] = (x\partial + q)\,\phi(x) \tag{22}$$

$$\frac{\partial\pi(x,u)}{\partial u} = -i[L_q, \phi(x)] = (x\partial + 1 - q)\,\pi(x) \tag{23}$$

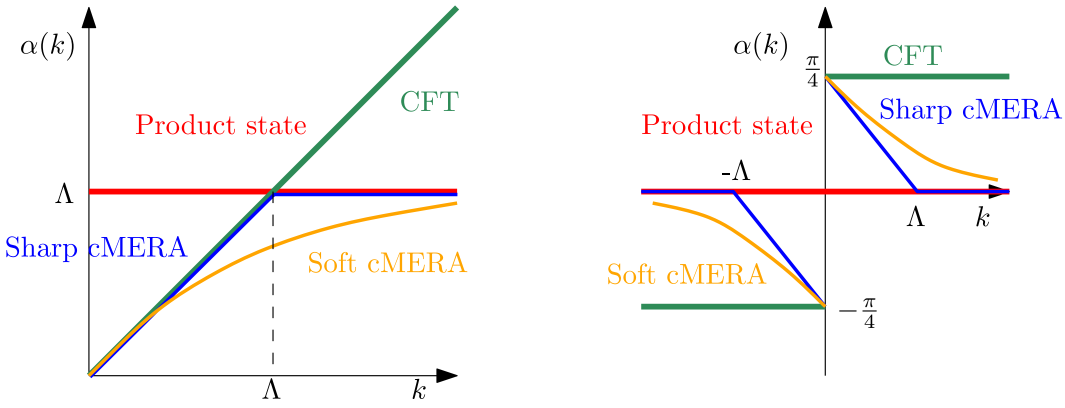

Figure 4: Qualitative drawings of $\alpha(k)$ for bosons (left) and fermions (right).

where we have defined $\phi(x,u) = e^{-iuL}\phi(x)e^{iuL}$ and $\pi(x,u) = e^{-iuL}\pi(x)e^{iuL}$. Now, from (23) it follows

$$\phi(x,u) = e^{uq}\phi(e^u x) \qquad \pi(x,u) = e^{u(1-q)}\pi(e^u x) \tag{24}$$

For our cMERA we will chose $q = 1/2$ so that, under $L = L_{1/2}$, ϕ and π have the same scaling dimensions. For the entangler K, we follow the proposal of [7], and set:

$$K = \frac{1}{2}\int_{-\infty}^{\infty} dk\, g(k)(\phi(k)\pi(-k) + h.c.) \tag{25}$$

where $g(k)$ is a function that incorporates the cutoff Λ, and has the general form:

$$g(k) = \frac{1}{2}\Gamma\left(\frac{k}{\Lambda}\right) \tag{26}$$

for some sufficiently fast decaying function Γ. In the case of the sharp regularization scheme in $\alpha(k)$ that we exposed before, Γ takes the form:

$$\Gamma(\kappa) = \Theta(1 - |\kappa|) \tag{27}$$

where Θ is the Heaviside function. We will soon present the reason why we say that these choices of $\alpha(k)$ and $g(k)$ are in some way compatible. But before let us note something interesting that happens when we write L_q in momentum space:

$$L_q = \frac{1}{2}\int_{-\infty}^{\infty} (\pi(-k)k\partial_k\phi(k) + k\partial_k\phi(k)\pi(-k) + (1-q)(\phi(k)\pi(-k) + \pi(-k)\phi(k))) \tag{28}$$

and compute $L+K$:

$$L+K = \frac{1}{2}\int_{-\infty}^{\infty}\left(\pi(-k)k\partial_k\phi(k) + k\partial_k\phi(k)\pi(-k) + \left(\frac{1}{2}+g(k)\right)(\phi(k)\pi(-k)+\pi(-k)\phi(k))\right) \tag{29}$$

That is, for $|k| > \Lambda$, $g(k) = 0$ and the integrand is that of L (the contribution from K is trivial). This we will call the nonrelativistic scaling operator, since ϕ and π have the same scaling dimensions, even though one is the time derivative of the other. This clearly breaks the relativistic principle of treating time on equal grounds with spatial coordinates. On the other hand, for $|k| \leq \Lambda$, the integrand is that of $L' = L_0$, which we will call the relativistic scaling operator, since now the scaling dimensions of ϕ and π are the correct ones required by Lorentz invariance. The fact that L is the scaling operator with respect to which the product state $|\Omega\rangle$ is invariant, and that L' is the one with respect to which the ground state of the CFT $|0\rangle$ is invariant adds then to the consistency of the construction, and begs as well the question: is there a scaling operator such that $|\psi\rangle$, our cMERA state, is invariant? The answer is yes, and as we anticipated before this will be the very same $L+K$ we have been constructing. Indeed, as long as the compatibility condition

$$\frac{\partial\alpha(k)}{\partial k} = \frac{2g(k)\alpha(k)}{k} \tag{30}$$

is satisfied, it can be proved (by expanding the exponential up to the linear term):

$$a(k)e^{-i(K+L)\delta u}\,|\psi\rangle = 0 \tag{31}$$

hence the state obtained from $|\psi\rangle$ by an infinitesimal scale transformation generated by $L+K$ belongs to the same common kernel of annihilation operators than $|\psi\rangle$ and is therefore, up to possibly a complex phase, the same state.

At this point we have to bring our attention to a particular point that we have ignored before: the real space structure of the entangler K. If we Fourier transform back to real space, we get

$$K = \frac{1}{2}\int_{-\infty}^{\infty} dx\,dy\,[\mu(x-y)\phi(x)\pi(y) + h.c.] \tag{32}$$

where

$$\mu(x-y) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dk\,g(k)e^{ik(x-y)} \tag{33}$$

We observe that in real space the entangler is nonlocal. How much it is so will be determined by the shape of $\mu(x)$, the Fourier transform of $g(k)$. If we choose the sharp cutoff $g(k)$, $\mu(x)$ turns out to be a sinc function:

$$\mu(x) = \frac{\sin(\Lambda x)}{x} = \Lambda\mathrm{sinc}(\Lambda x) \tag{34}$$

which decays as a power law. It is nevertheless preferable to have the entangler decay a least exponentially to retrieve some notion of locality. First of all, this brings an analogy with the lattice, where the entangling operations have a characteristic length scale, namely the lattice spacing. Additionally, it is expected to simplify things when we move on to interacting theories where the interaction terms are local. A good way to achieve it is to soften the cutoff to a Gaussian profile (*soft regularization scheme*, see figure 4):

$$\Gamma(\kappa) = e^{-\kappa^2} \implies \mu(x) = \frac{\Lambda}{4\sqrt{\pi}} e^{-\frac{(\Lambda x)^2}{4}} \tag{35}$$

The functions $g(k)$ and $\alpha(k)$ end up being then

$$g(k) = \frac{1}{2} e^{-\frac{k^2}{\Lambda^2}} \qquad \alpha(k) = \mathrm{Expi}\left(-\frac{1}{2}\frac{k^2}{\Lambda^2} e^{-\gamma}\right) \tag{36}$$

where Expi is the exponential integral function, and γ is the Euler-Mascheroni constant.

2.4 A second example: 1+1 free Dirac fermions

Let's apply the same procedure now to a fermionic theory. The Hamiltonian for a couple of massless, spinless Dirac fermions ψ_1, ψ_2 is given by

$$H = -i \int_{-\infty}^{\infty} dx \, (\psi_1(x)^\dagger \partial_x \psi_2(x) - \psi_2(x)^\dagger \partial_x \psi_1(x)) \tag{37}$$

Through Fourier transformation it can be recast in the form

$$H = \int_{-\infty}^{\infty} dk \, k \left(\psi_1(k)^\dagger \psi_2(k) + \psi_2(k)^\dagger \psi_1(k)\right) = \int_{-\infty}^{\infty} dk \, |k| \left(\tilde{\psi}_1(k)^\dagger \tilde{\psi}_1(k) - \tilde{\psi}_2(k)^\dagger \tilde{\psi}_2(k)\right) \tag{38}$$

where we have defined the rotated fermionic variables

$$\tilde{\psi}_1(k) = \frac{\psi_1(k) + \mathrm{sign}(k)\psi_2(k)}{\sqrt{2}} \qquad \tilde{\psi}_2(k) = \frac{-\mathrm{sign}(k)\psi_1(k) + \psi_2(k)}{\sqrt{2}} \tag{39}$$

Given the last expression for the Hamiltonian it is clear that its ground state will be given by:

$$\tilde{\psi}_1(k) \, |0\rangle = 0 \text{ and } \tilde{\psi}_2(k)^\dagger \, |0\rangle = 0 \tag{40}$$

i.e., in the ground state the modes which have positive energy are unoccupied (thus the corresponding annihilation operator annihilates the vacuum), and those with negative energy are occupied (thus, the corresponding creation operator annihilates the vacuum, since we are speaking about fermions). Note that there is an ambiguity in the case of the zero modes (corresponding in our case to $k = 0$). This leads to a degenerate ground state. Our cMERA state will look, at small momenta, like one of these ground states, which we will chose arbitrarily in section 3, when we start making computations with this theory, as the one given by the convention sign(0)=1. The product state can be obtained, just as we did before, as the continuous limit of a product state in a fermion lattice. It will be more natural for us to then consider ψ_2 as involving particles of negative energy (antiparticles), whose modes will hence be occupied in the vacuum state $|\Omega\rangle$:

$$\psi_1(x)\,|\Omega\rangle = 0 \text{ and } \psi_2(x)^\dagger\,|\Omega\rangle = 0 \tag{41}$$

or

$$\psi_1(k)\,|\Omega\rangle = 0 \text{ and } \psi_2(k)^\dagger\,|\Omega\rangle = 0 \tag{42}$$

Note that this is the ground state of the following Hamiltonian:

$$H = \int_{-\infty}^{\infty} dk\,|k|\left(\psi_1(k)^\dagger\psi_1(k) - \psi_2(k)^\dagger\psi_2(k)\right) \tag{43}$$

The key realization here is that both Hamiltonians, and hence both ground states differ only by a rotation of the fields involved. We define our cMERA state $|\psi\rangle$ as follows:

$$(\cos(\alpha(k))\psi_1(k) + \sin(\alpha(k))\psi_2(k))\,|\psi\rangle = 0 \tag{44}$$

$$(-\sin(\alpha(k))\psi_1(k)^\dagger + \cos(\alpha(k))\psi_2(k)^\dagger)\,|\psi\rangle = 0 \tag{45}$$

And from now on, we denote

$$\tilde{\psi}_1(k) = \cos(\alpha(k))\psi_1(k) + \sin(\alpha(k))\psi_2(k) \tag{46}$$

$$\tilde{\psi}_2(k) = -\sin(\alpha(k))\psi_1(k) + \cos(\alpha(k))\psi_2(k) \tag{47}$$

$|\psi\rangle$ will be of course the ground state of the Hamiltonian

$$H = \int_{-\infty}^{\infty} dk\,|k|\left(\tilde{\psi}_1(k)^\dagger\tilde{\psi}_1(k) - \tilde{\psi}_2(k)^\dagger\tilde{\psi}_2(k)\right) \tag{48}$$

Note that both the vacuum state and the product state are of this form, with $\alpha(k) = \text{sign}(k)\frac{\pi}{4}$ and $\alpha(k) = 0$ respectively. Hence our first choice of $\alpha(k)$ for the cMERA would be (sharp regularization scheme, see figure 4):

$$\alpha(k) = \Theta(\Lambda - |k|)\text{sign}(k)\frac{\pi}{4}\left(1 - \frac{|k|}{\Lambda}\right) \tag{49}$$

Just as before we can find a scaling operator L and an entangler K:

$$L = \frac{-i}{2}\int_{-\infty}^{\infty} dx \sum_{\alpha=1,2}\Big(\psi_\alpha(x)^\dagger x\partial\psi_\alpha(x) - x\partial\psi_\alpha(x)^\dagger\psi_\alpha(x)\Big) = \tag{50}$$

$$= i\int_{-\infty}^{\infty} dk \sum_{\alpha=1,2}\left(\psi_\alpha(k)^\dagger k\partial\psi_\alpha(k) + \frac{1}{2}\psi_\alpha(k)^\dagger\psi_\alpha(k)\right) \tag{51}$$

$$K = i\int_{-\infty}^{\infty} dk\, g(k)[\psi_1(k)^\dagger\psi_2(k) - \psi_2(k)^\dagger\psi_1(k)] \tag{52}$$

as well as a compatibility condition between $\alpha(k)$ and $g(k)$ for the cMERA to be invariant under $L+K$:

$$\frac{\partial\alpha(k)}{dk} = -\frac{g(k)}{k} \tag{53}$$

Also as it happened for the bosons, when we use the sharp regularization scheme it turns out that the entangler K is non local in real space. Indeed,

$$K = -i\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dx\, dy\, \mu(x-y)[\psi_1(x)^\dagger\psi_2(y) - \psi_2(x)^\dagger\psi_1(y)] \tag{54}$$

where

$$\mu(x-y) = \int_{-\infty}^{\infty} dk\, e^{-ik(y-x)}g(k) \tag{55}$$

gives again an idea of the how non-local the entangling procedure is. For the theory with a sharp regularization we have

$$\Gamma(\kappa) = \Theta(1-|\kappa|) \qquad g(k) = \frac{\pi}{4}\frac{k}{\Lambda}\Gamma\left(\frac{k}{\Lambda}\right) \qquad \mu(x) = \frac{i\Lambda\pi}{2}\frac{\Lambda x\cos\Lambda x - \sin\Lambda x}{(\Lambda x)^2} \tag{56}$$

where again μ presents a power law decay, so we might consider changing to a soft Gaussian regularization, for which (see figure 4):

$$\Gamma(\kappa) = \frac{2}{\sqrt{\pi}}e^{-\kappa^2} \qquad g(k) = \frac{\pi}{4}\frac{k}{\Lambda}\Gamma\left(\frac{k}{\Lambda}\right) \qquad \alpha(k) = \frac{\pi}{4}\left(1-\mathrm{erf}\left(\frac{k}{\Lambda}\right)\right) \tag{57}$$

$$\mu(x) = \frac{-i\Lambda^2\pi}{4}xe^{-\frac{(\Lambda x)^2}{4}} \tag{58}$$

3 Correlations and entanglement profile of a cMERA state

After so much exposition, it is time for us to take action and start working with our new toolset. Our aim in this section will be to characterize the entanglement properties of a cMERA ground state. We will do this by computing and analyzing:

i) the correlation functions (two-point functions) for the operators of the theory, and ii) the entanglement profile of the cMERA state, i.e. the scaling of the entanglement entropy of a region with its size. We will also take a look at the shape of the entanglement contours. As it was exposed above, the study of entanglement properties has proved itself very useful in quantum many-body physics to extract information from quantum states, and in particular we expect to gain insight about whether cMERA achieves its goal of "interpolating" between the ground state of a CFT and a product state, and in which way it is achieved. Along this section, we will explain the procedures followed in full generality, and we will apply them to a particular cMERA: the one built from the 1+1 Dirac fermion CFT presented in section 2.4. We will work in parallel with the sharp and soft regularizations schemes, so that we are able to compare them. Let's start!

3.1 Correlation functions

Two-point correlators are easy to obtain for the cMERA, given its characterization in terms of creation-annihilation operators and the (anti)-commutation algebra they satisfy. Fortunately for us, the cMERA states we are going to care about satisfy a particular property that makes these two-point correlators extremely important: they are **Gaussian states**. A Gaussian state is, to give a short definition, one such that Wick's theorem (factorization of the N-point function into two-point correlators) holds:

$$\langle \mathcal{O}_1 \dots \mathcal{O}_N \rangle = \sum_{\text{pairings}} \langle \mathcal{O}_{i_1} \mathcal{O}_{i_2} \rangle \dots \langle \mathcal{O}_{i_{N-1}} \mathcal{O}_{i_N} \rangle \tag{59}$$

where the sum is taken over all possible pairings of the operators of the N-point function we want to compute. It turns out that ground states of quadratic Hamiltonians are always Gaussian states, and this applies in particular to cMERA states (see (20) and (48)). For these states, the two-point correlators acquire special relevance since they contain enough information to completely determine the state. To work with them easily, we arrange the correlators in the so-called **correlation matrix**:

$$M^{|\psi\rangle}_{\alpha,\beta}(x,y) = \langle \mathcal{O}_\alpha(x) \mathcal{O}_\beta(y) \rangle \tag{60}$$

where the $\mathcal{O}_\alpha$ stand for the corresponding operators of the theory (ϕ and π for bosonic theories, ψ and $\psi^\dagger$ for fermionic ones). From now on, we will omit the specification of with respect to which state the correlation matrix is taken, whenever this is clear.

Particularizing to our 1-dimensional fermions, the two-point function in momentum space $\langle\psi_1^\dagger(p)\psi_1(p)\rangle$ can be computed as follows:

$$\begin{aligned}&\langle\psi_1^\dagger(p)\psi_1(q)\rangle =\\&= \langle(\cos(\alpha(p))\tilde{\psi}_1(p) - \sin(\alpha(p))\tilde{\psi}_2(p))^\dagger(\cos(\alpha(q))\tilde{\psi}_1(q) - \sin(\alpha(q))\tilde{\psi}_2(q))\rangle =\\&= \sin(\alpha(p))\sin(\alpha(q))\langle\tilde{\psi}_2(p))^\dagger\tilde{\psi}_2(q)\rangle = \sin^2(\alpha(p))\delta(p-q) \qquad (61)\end{aligned}$$

where in the last step we have used that canonical anticommutation relations are preserved by unitaries such as the rotation that defines $\tilde{\psi}_1, \tilde{\psi}_2$. In the same fashion we can obtain

$$\langle\psi_1^\dagger(p)\psi_2(q)\rangle = -\frac{1}{2}\sin(2\alpha(p))\delta(p-q) = \langle\psi_2^\dagger(p)\psi_1(q)\rangle \qquad (62)$$

$$\langle\psi_2^\dagger(p)\psi_2(q)\rangle = \cos^2(\alpha(p))\delta(p-q) = \delta(p-q) - \langle\psi_1^\dagger(p)\psi_1(q)\rangle \qquad (63)$$

The correlators in real space, such as $\langle\psi_1^\dagger(x)\psi_1(y)\rangle$ are now just a Fourier transform away from us. For the sharp regularization scheme the result can be obtained analytically as shown below:

$$\begin{aligned}\langle\psi_1^\dagger(x)\psi_1(y)\rangle &= \int\int \frac{dp\,dq}{2\pi}e^{-i(px-qy)}\sin^2(\alpha(p))\delta(p-q) =\\&= \int_{-\Lambda}^{\Lambda}\frac{dp}{2\pi}e^{-ip(x-y)}\sin^2\left(\mathrm{sign}(p)\frac{\pi}{4}\left(1-\frac{|p|}{\Lambda}\right)\right) =\\&= \frac{\pi\sin(\Lambda(x-y)) - 2\Lambda(x-y)}{2(x-y)\left(\pi^2 - 4\Lambda^2(x-y)^2\right)} \qquad (64)\end{aligned}$$

And again in the same manner,

$$\langle\psi_1^\dagger(x)\psi_2(y)\rangle = \frac{i\Lambda(\pi\sin(\Lambda(x-y)) - 2\Lambda(x-y))}{\pi^3 - 4\pi\Lambda^2(x-y)^2} \qquad (65)$$

$$\langle\psi_2^\dagger(x)\psi_2(y)\rangle = \delta(x-y) - \frac{\pi\sin(\Lambda(x-y)) - 2\Lambda(x-y)}{2(x-y)\left(\pi^2 - 4\Lambda^2(x-y)^2\right)} = \delta(x-y) - \langle\psi_1^\dagger(x)\psi_1(y)\rangle \qquad (66)$$

On the other hand, for the soft regularization scheme the real space correlators are most easily determined numerically. Both cases are represented in figures 5 and 6, in units where $\Lambda = 1$. Notice that the functions $\langle\psi_1^\dagger(0)\psi_1(x)\rangle$ and $\langle\psi_1^\dagger(0)\psi_2(x)\rangle$ are all we need to know since i) correlators are a function only of the distance between the two points (cMERA is translation invariant), and ii) the $\psi_2^\dagger\psi_2$ correlator can easily be written in terms of the $\psi_1^\dagger\psi_1$ correlator. Qualitatively, the behaviour of correlators is very similar independently of the regularization scheme. The most visible differences at this respect can be appreciated in the regions near $\Lambda x \sim 1$, where the sharply regularized state presents some oscillations (which remind us of the sinc function in the entangler for this case) that the softly regularized state does

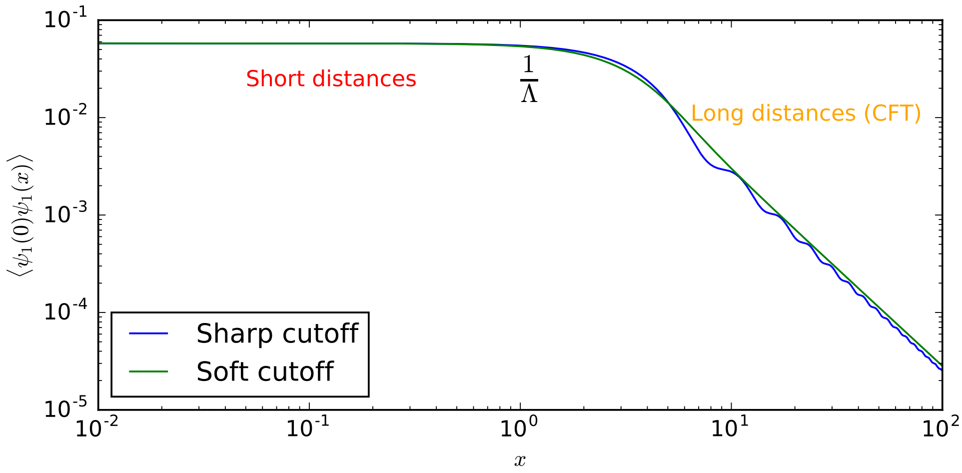

Figure 5: $\langle\psi_1^\dagger(0)\psi_1(x)\rangle$

not. In the limits $\Lambda x \ll 1$ and $\Lambda x \gg 1$, the two graphs become parallel, telling us that the only difference between both two-point functions is a multiplicative constant close to 1. Now for the general analysis of what these figures mean: they reflect the existence of two very different regimes for the correlators. The correlator $\langle\psi_1^\dagger(0)\psi_1(x)\rangle$ is approximately constant and flat until it reaches an inflection point at $x\Lambda \sim O(1)$, i.e., when the distance between points reaches the cutoff length scale of the theory, $1/\Lambda$. Past this point, the correlator decreases as a (quadratic) power law:

$$\langle\psi_1^\dagger(0)\psi_1(x)\rangle \sim \begin{cases} C & \Lambda x \ll 1 \\ \frac{1}{x^2} & \Lambda x \gg 1 \end{cases} \tag{67}$$

As for the correlator $\langle\psi_1^\dagger(0)\psi_2(x)\rangle$, its absolute value (notice that it is purely imaginary) grows linearly from 0 at $x = 0$ while $\Lambda x \ll 1$, reaches a maximum at distances of order of the cutoff lengthscale, and subsequently decays as a power law[5]:

$$\langle\psi_1^\dagger(0)\psi_2(x)\rangle \sim \begin{cases} x & \Lambda x \ll 1 \\ \frac{1}{x} & \Lambda x \gg 1 \end{cases} \tag{68}$$

[5]In the sharp cutoff scheme, we have indeed a very simple relation between the two correlation functions:

$$\langle\psi_1^\dagger(x)\psi_2(y)\rangle = \frac{2i\Lambda(x-y)}{\pi}\langle\psi_1^\dagger(x)\psi_1(y)\rangle$$

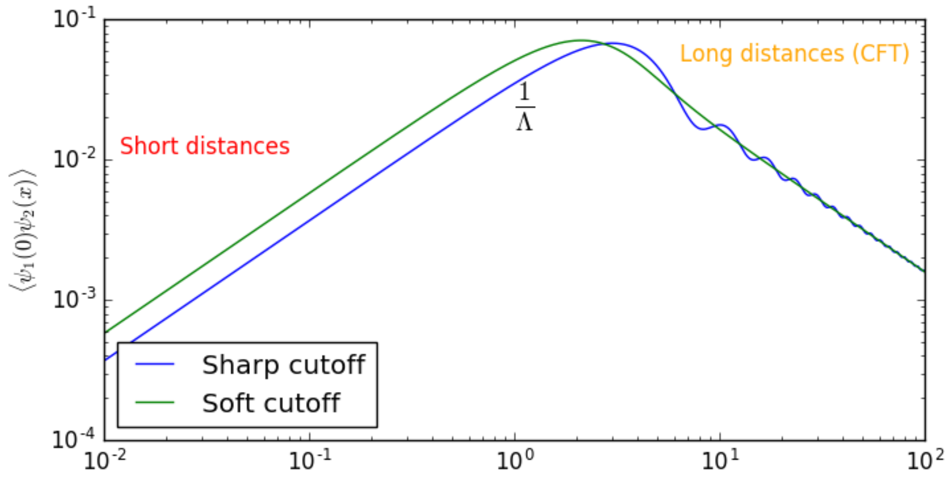

Figure 6: $|\langle\psi_1^\dagger(0)\psi_2(x)\rangle|$

In summary, correlations in the cMERA behave differently depending on how the length scale at which we look compares with the cutoff given by $1/\Lambda$. The large distance regime shows the expected features of the ground state of a CFT, namely a power law decay of correlations (remember our discussion of MERA at the beginning of section 2). At short distances, however, the intrinsic cutoff of the theory prevents the correlations from diverging (save, of course, from the on-site delta correlation of each fermionic species with itself), even reducing the $\psi_1^\dagger\psi_2$ correlator to 0. This hints at the success of these regularization schemes in producing a state that reproduces a CFT ground state up to a certain scale. Let us now move to the entanglement entropy: hopefully the results there will support our conclusions from this first part.

3.2 Entanglement entropy

Suppose that we choose some spatial subregion $\mathcal{R}$ of our system and trace out the rest. The reduced state on the Hilbert space associated with $\mathcal{R}$ will still be Gaussian. Indeed, Wick's theorem for the local algebra of operators supported on $\mathcal{R}$ is just a particular case of Wick's theorem for the total state. Furthermore, the correlation matrix that characterizes the reduced state is given by the restriction of the spatial index of the original correlation matrix to $\mathcal{R}$:

$$\rho_{\mathcal{R}} = \mathrm{tr}_{\overline{\mathcal{R}}}(|\psi\rangle\langle\psi|) \implies M_{\mathcal{R}} = M^{\rho}_{\alpha,\beta}(x,y) = M^{|\psi\rangle}_{\alpha,\beta}(x,y)\Big|_{x,y\in\mathcal{R}} \tag{69}$$

All the information of the reduced density matrix $\rho_{\mathcal{R}}$ is contained in the correlation matrix $M_{\mathcal{R}}$, in particular its von Neumann entanglement entropy $S(\rho_{\mathcal{R}}) = -\text{tr}\rho_{\mathcal{R}} \log_2 \rho_{\mathcal{R}}$. Indeed, it can be obtained directly from $M_{\mathcal{R}}$, provided that we can build a set of uncorrelated modes from the modes localized in $\mathcal{R}$. This translates into the correlation matrix for $\mathcal{R}$ being diagonal, and the density matrix decomposing into a tensor product on these modes, what in turns simplifies the computation of the entropy:

$$\rho = \bigotimes_i \rho_i \implies S(\rho) = \sum_i S(\rho_i) \tag{70}$$

For bosonic variables, this computation results in the expression:

$$S(\rho) = \sum_\lambda \left[-\left(\lambda - \frac{1}{2}\right) \log_2 \left(\lambda - \frac{1}{2}\right) + \left(\lambda + \frac{1}{2}\right) \log_2 \left(\lambda + \frac{1}{2}\right) \right] \tag{71}$$

where the sum runs over the *symplectic eigenvalues* of the correlation matrix. For fermionic variables, the corresponding result is

$$S(\rho) = -\sum_\lambda \left[\lambda \log_2 \lambda + (1-\lambda) \log_2 (1-\lambda)\right] \tag{72}$$

and the sum runs over the (standard) eigenvalues of the correlation matrix [9]. These are hence the ones we want to compute in this section for our fermionic cMERA. Let us first wonder about the short distance regime, and how much it displays properties akin to those of a product state. For intervals of length L much shorter than $1/\Lambda$, the correlation matrix of the reduced state turns out to be quite simple, in the sense that it can be well approximated by a few initial terms of its Taylor expansion around zero:

$$M(x, y) = M_0 + xM_{1x} + yM_{1y} + \dots \tag{73}$$

The precise determination of how much the computed value of the entropy is affected by this approximation is an interesting problem in perturbation theory that exceeds nevertheless the scope of this essay (hence we do not aim to be very rigorous in this part). A perturbation of order $O(\epsilon)$ of the eigenvalues of the matrix M might result in a change in their contribution to the entropy of order up to $O(\epsilon \log \epsilon)$ since the function $-x \log_2(x) - (1-x) \log_2(x)$ that gives the entropy in terms of the said eigenvalues is not analytic at 0 or 1, which will be a very common value for the eigenvalues of the approximate matrix. Fortunately, a very low order expansion will already give us very good precision in the computation of the entropy, as we will see in a few pages. Let us then assume $x, y \ll 1\Lambda$, and approximate our correlation matrix by the first term of its Taylor expansion. To keep things simple, we will do it first for the sharp regularization scheme and we will see what we can infer for the soft one afterwards.

$$M_{\alpha\beta}(x,y) = \langle\psi^{\dagger}_{\alpha}(x)\psi_{\beta}(y)\rangle \approx \begin{cases} c_1 & \alpha=\beta=1 \\ c_2(x-y) & \alpha\neq\beta \\ \delta(x-y)-c_1 & \alpha=\beta=2 \end{cases} \tag{74}$$

where c_1 and c_2 are constants. Let us now study such a matrix's spectrum. We can prove that all of its eigenvalues will be 1 or 0 save for at most four of them. Remember that these eigenvalues provide us with the entanglement entropy of the system, and that the contribution from either a 1 or a 0 to the entropy vanishes (it represents a mode in a pure state $|1\rangle\langle 1|$ or $|0\rangle\langle 0|$ which displays no mixing). Hence our result hints at the fact that no matter how much we increase the size of our interval, our entanglement entropy will be bounded by the maximal contribution of this four modes (namely 4 entanglement bits or *ebits*). Of course, only as long as the linear approximation holds, otherwise we will have to take more terms into account in the expansion, and the number of modes that contribute to the entropy will increase. To prove our statement consider an eigenvector $f_\alpha(x)$ of $M_{\alpha\beta}(x,y)$ of eigenvalue λ. Here we consider the correlation matrix restricted to some region of space $[0,L]$ with $L \ll \dfrac{1}{\Lambda}$. This means

$$\int_0^L M_{\alpha\beta}(x,y) f_\beta(y)\,dy = \lambda f_\alpha(x) \implies \tag{75}$$

$$c_1\int_0^L f_1(y)\,dy + c_2\int_0^L (x-y)f_2(y)\,dy = \lambda f_1(x) \tag{76}$$

$$c_2\int_0^L (x-y)f_1(y)\,dy + \int_0^L \delta(x-y)f_2(y)\,dy - c_1\int_0^L f_2(y)\,dy = \lambda f_2(x) \tag{77}$$

Rewriting the last equation as

$$c_2\int_0^L (x-y)f_1(y)\,dy - c_1\int_0^L f_2(y)\,dy = (1-\lambda)f_2(x) \tag{78}$$

it turns out that if $\lambda \neq 0,1$, then we must have f_1 and f_2 to be linear functions:

$$f_1(x) = a + bx \qquad f_2(x) = c + dx \tag{79}$$

whose coefficients also satisfy

$$\begin{aligned} \lambda a &= c_1\int_0^L f_1(y)\,dy - c_2\int_0^L y f_2(y)\,dx & \lambda b &= c_2\int_0^L f_2(y)\,dx \\ (\lambda-1)c &= c_1\int_0^L f_2(y)\,dy - c_2\int_0^L y f_1(y)\,dx & (\lambda-1)d &= c_2\int_0^L f_1(y)\,dy \end{aligned}$$

This leads us to λ belonging to the eigenvalues of the following, four dimensional matrix:

$$A = \begin{pmatrix} c_1 L & \frac{c_1 L^2}{2} & -\frac{c_2 L^2}{2} & -\frac{c_2 L^3}{3} \\ 0 & 0 & c_2 L & \frac{c_2 L^2}{2} \\ -\frac{c_2 L^2}{2} & -\frac{c_2 L^3}{3} & 1 - c_1 L & -\frac{c_1 L^2}{2} \\ c_2 L & \frac{c_2 L^2}{2} & 0 & 1 \end{pmatrix} \tag{80}$$

Once we substitute the values of the constants $c_1 = \frac{(\pi-2)\Lambda}{2\pi^2}$ and $c_2 = \frac{i(\pi-2)\Lambda^2}{\pi^3}$ (which correspond to the sharp regularization scheme) and diagonalize the matrix, it turns out that the eigenvalues only depend on ΛL (i.e. on the ratio between the length scales given by L and $1/\Lambda$), and since we are in the regime where this is a small number we may expand them in a power series:

$$\begin{aligned} \lambda_1 &= \frac{(\pi-2)\Lambda L}{2\pi^2} + \frac{\left(-4+4\pi-\pi^2\right)\Lambda^4 L^4}{12\pi^6} + O\left(\Lambda^5 L^5\right) \\ \lambda_2 &= 1 + \left(\frac{1}{\pi^2} - \frac{1}{2\pi}\right)\Lambda L + \frac{\left(4-4\pi+\pi^2\right)\Lambda^4 L^4}{12\pi^6} + O\left(\Lambda^5 L^5\right) \\ \lambda_3 &= 1 + \frac{\left(4-4\pi+\pi^2\right)\Lambda^4 L^4}{12\pi^6} + O\left(\Lambda^5 L^5\right) \\ \lambda_4 &= \quad -\frac{\left(4-4\pi+\pi^2\right)\Lambda^4 L^4}{12\pi^6} + O\left(\Lambda^5 L^5\right) \end{aligned} \tag{81}$$

Of these four eigenvalues, two (λ_1 and λ_2) are indeed between 0 and 1 and provide the only nontrivial contribution to the entropy in this approximation, which we compute to be:

$$S(L) = \frac{\pi-2}{\pi^2 \ln 2}\left(1 + \ln\left(\frac{2\pi^2}{\pi-2}\right)\right)\Lambda L - \frac{\pi-2}{\pi^2 \ln 2}\Lambda L \ln \Lambda L + \mathrm{O}(\Lambda^2 L^2 \ln \Lambda L) \tag{82}$$

What happens with the other two eigenvalues, λ_3 and λ_4? They cannot provide an entropy contribution because they are outside the interval $[0, 1]$ (though very close to its boundary). It turns out that these eigenvalues are an artifact of our first-order approximation of the matrix, and it can be checked that they get closer to 0 and 1 as we add more terms. Also, notice their contribution is of order $O(L^4\Lambda^4)$, but to get the full contribution at that order we should at least have expanded the matrix up to that order.

What happens if we use the soft regularization scheme? The value of c_1 is to a very good approximation the same as in the sharp case, while c_2 changes appreciably. Nevertheless, solving for the entropy's series expansion directly in terms of c_1 and c_2 shows that c_2 only contributes from order $O(\Lambda^4 L^4 \log \Lambda L)$ on, hence the first terms of the expansion for the entropy are essentially equal for both schemes. Summing up, $S(L)$ tends to zero when the size of the interval shrinks, and we will see that the two leading order terms given in (82) approximate it very well until $\Lambda L \sim 1$, when the approximation used is no longer valid.

Let us proceed to numerically compute the entropy scaling for the fermionic cMERA state. To do it we have to induce a discretization at the level of the correlation matrix in real space, after which we compute its eigenvalues and plot the entropy. Figure 7 shows our results for the entanglement entropy for the sharp and soft regularization.

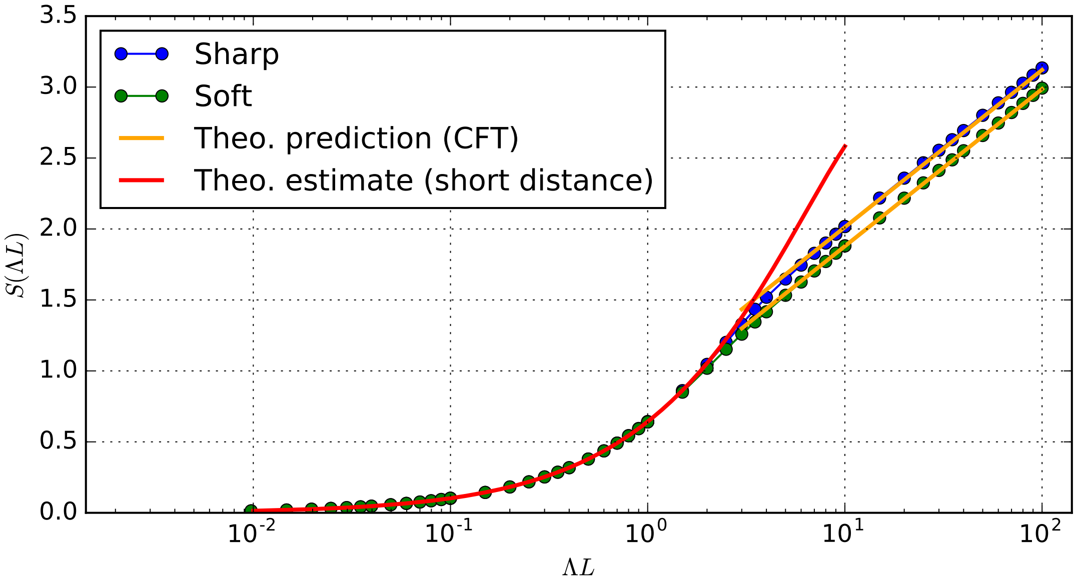

Figure 7: Numerically determined entanglement entropy profile and theoretical predictions in short and long distance regimes, given for both sharp and soft regularization schemes.

How stable are this results with respect to the discretization lattice spacing a? The answer is, they stabilize very quickly. Figure 8 shows the behaviour of the difference between the computed entropy for $\Lambda L = 64$ and our best approximation to its real value[6] (the result of the computation with the finest graining) with the lattice spacing a. We obtain for both regularizations an almost straight line of slope ~ -2.1, thus we deduce that the error decays approximately with a^2 upon fine graining. In particular, this means that the entanglement entropy has converged to a finite value, something that does not happen when we study the original CFT. Indeed, in the full CFT, the entanglement entropy for a finite region diverges, since infinitely many high energy modes contribute to it. The only way in which we can plot a finite entropy

[6]The choice of that value of ΛL is due to the fact that for larger values of it the convergence is slower, and the differences are easier to appreciate.

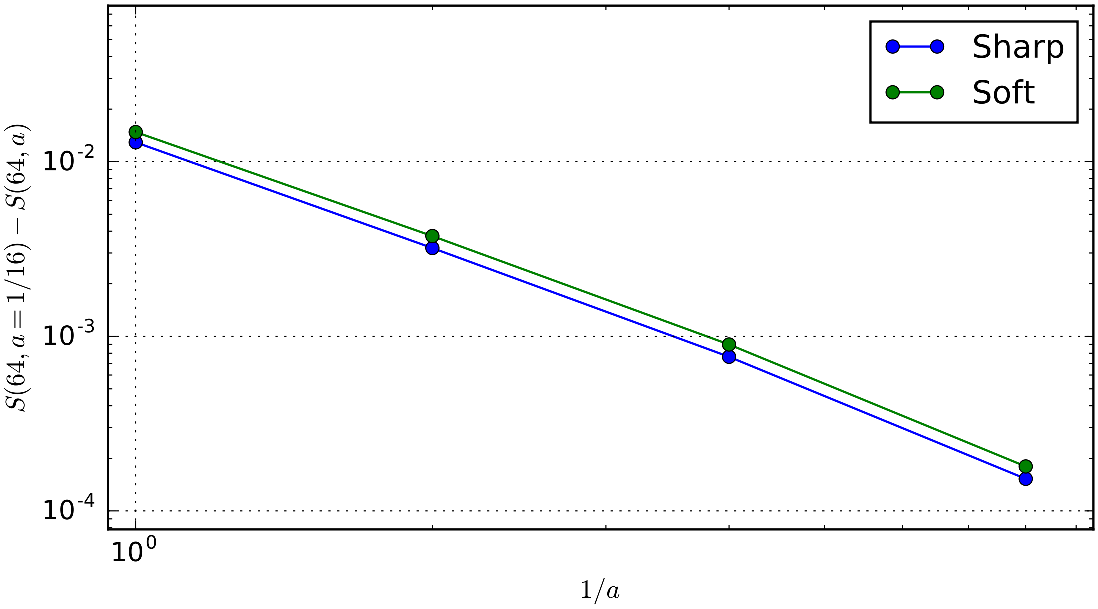

Figure 8: The order of convergence is approximately quadratic.

is by introducing a UV cutoff, which is what happens when we place our theory in a lattice with lattice spacing a, which has a cutoff at $\frac{2\pi}{a}$. Nevertheless, sending the lattice spacing back to zero results in this cutoff diverging and so does the entropy as well. The cMERA, on the other hand, has an intrinsic cutoff at $k = \Lambda$. Once the lattice cutoff goes past it, the intrinsic cutoff becomes the most restrictive one, and it stays constant, so that no new modes enter to give contributions to the entropy. Going back to figure 7, it is patent that both theoretical predictions are satisfied by the data to a very good extent. On the one hand, the expression in equation (82) explains the entropies for both regularization schemes up to $\Lambda L \sim 1$. In fact, the transition seems to be close to $\Lambda L \approx 3$, which is the point at which we observed it for the correlators. Note also that it is seemingly smoother for the soft regularization than for the sharp one. On the other hand, for the CFT (large distance) region we expect a logarithmic scaling of entropy, with some slope proportional to the central charge c of the theory

$$S(L) \sim \frac{c}{3} \log L \tag{83}$$

and indeed this turns out to be the case. Each fermionic species contributes $1/2$ to the central charge, hence we require a value of $c = 1$. Linear fits of the results in this region provide the values

$$c_{\text{sharp}} = 1.007 \qquad c_{\text{soft}} = 1.003 \tag{84}$$

which are in very good agreement with our expectation.

Entanglement contours

In our procedure of computation of entanglement entropy we ended up with a set of uncorrelated modes χ_n, each of which gave us a precise contribution to the total entropy $s(\chi_n)$. If we now perform a (unitary) linear transformation in the space of modes to get a different basis ξ_m, there is a way in which we can assign entropy contributions $s(\xi_m)$ to these modes as well, so that the total entropy is the same:

$$\sum_n s(\chi_n) = \sum_m s(\xi_m) \tag{85}$$

Indeed, we can just assign them as follows. Let U_{mn} be the unitary that transforms one basis into the other, and define

$$\xi_m = U_{mn}\chi_n \implies s(\xi_m) = |U_{mn}|^2 s(\chi_n) \tag{86}$$

where the sum over repeated indices is implied. When we choose ξ_m to be a set of localized modes (like our $\psi_1(x), \psi_2(x)$), the entropy contributions give rise to the **entanglement contour**:

$$s(x) = s(\psi_1(x)) + s(\psi_2(x)) \tag{87}$$

which provides us with an intuition on how the degrees of freedom in different positions contribute to the entanglement. When we do these computations for the cMERA, we find that most of the entanglement contributions come from the boundary modes, as it was observed for standard quantum field theories [10]. We observe one peculiarity of cMERA entanglement contours, and that is that Λ seems to be present in them as well. Indeed, when we plot the contours (see figure 9), we see that the boundary accumulation of entropy seems to reach a fixed size that does not change when we increase the size of the interval, and is of the order to twice the length scale at which the transition between the two regimes happens, i.e., about $6/\Lambda$. Since that contribution is fixed, the scaling of entropy is given mainly by the central part of the contour, and this we observe to be compatible with $s(x) \sim 1/x$, which gives the correct logarithmic scaling.

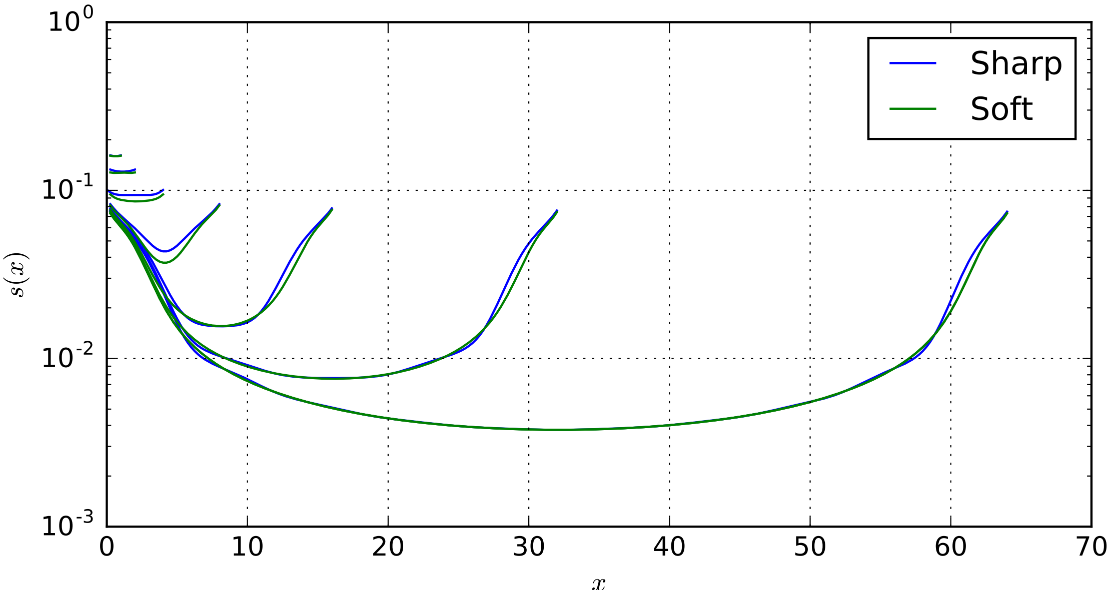

Figure 9: Entanglement contours for intervals of size $\Lambda L = 1, 2, 4, 8, 16, 32$ and 64, and lattice spacing $\Lambda a = 0.25$.

4 Interpretation and conclusions

In this essay we have first reviewed the continuous MERA [7], a construction that brings the philosophy of entanglement renormalization to the study of quantum field theories, and provides in this way a UV regularization procedure that sucesfully removes the entanglement contribution of high energy modes. We have then initiated the study of correlations and entanglement entropy in cMERA, focusing on a particular CFT. We found that the structure of these very patently interpolate between the CFT ground state at large distances and the product state at short distances.

4.1 Implications

This interpolation is accomplished while preserving the continuous nature of the system. That provides the main difference with the already well-known MERA techniques, which introduce a UV cutoff via lattice discretization. By remaining in the continuous setup, cMERA can keep symmetries such as traslation and rotation intact, rather than breaking them into discrete subgroups. We can also perform linearizations and work with the Lie algebras of generators of the symmetry groups. On a practical level, keeping a continuous system even when we regularize helps us avoid technical inconveniences that appear on the lattice: for example, fermion doubling. When placing a fermionic theory on the lattice, the number of fermionic species doubles with each discretized dimension. There exist several strategies to deal with this problem; cMERA does it by removing the need to discretize.

4.2 A note on the c-theorem

The magnitude that we have been studying in this essay, entanglement entropy, is no stranger to modern scientific literature, and it has been thoroughly studied in the context of quantum field theories (see [11, 12, 13] among many others).

In [11], Casini and Huerta give a proof of the c-theorem: the existence, for any 1+1 *relativistic* quantum field theory, of a c-function

$$c(r) = r\frac{dS(r)}{dr} \tag{88}$$

which is a universal dimensionless function which is nonincreasing under dilatations and takes a finite value proportional to the central charge for CFTs. Here $S(r)$ is the entanglement entropy of an interval of size r. The property of being nonincreasing:

$$c'(r) = rS''(r) + S'(r) \leq 0 \tag{89}$$

also implies that $S(r)$ is concave, i.e., the slope with which it increases is smaller for bigger r. That is, nevertheless, not what we observe for cMERA. Indeed, in figure 7 we notice that the slope of the entropy increases when we leave the product state regime, and enter the CFT. The reason for this apparent contradiction is very simple: cMERA is not a relativistic theory. Indeed, the product state we have been using is not Lorentz invariant, nor is the Hamiltonian for which cMERA is a ground state. This allows for the violation of the concavity of the entropy function, which assumes Lorentz invariance.

4.3 Future directions

There are many research projects for which the contents of this essay would provide a starting point. One obvious aspect that is missing is the confirmation of whether the results presented extend to higher dimensional free CFTs. It is to be expected that it will be the case since the generalization process does not involve any nonstandard manipulation. At the moment of submission of this essay we are working in repeating our results for a bosonic theory in 2 dimensions.

Currently there are efforts being done in trying to extract conformal data from the cMERA. It is well known that a CFT is characterized by its set of primary operators, and its conformal dimensions and spins. Preliminary results have shown that we can recover the conformal towers with the right scaling dimensions from the original CFT in the context of the cMERA. This time, the scaling operators are given by smeared versions of the fields of the theory.

Another research line that currently involves cMERA is trying to realize the full conformal group on the cMERA. We have seen that the scale transformations need to be redefined in order for the cMERA state to remain scale invariant. Lorentz boosts, as we mentioned before, cannot be kept the way they are if we want them to leave the cMERA invariant. Translations and rotations, though, appear not to require any modification: cMERA is manifestly translation and rotation invariant with respect to the traditional definitions of these symmetries. It turns out that indeed it is possible to realize all of the elements of the conformal group on the cMERA, thus rendering it a CFT on its own [8].

But the true potential of cMERA promises to be realized in the context of interacting theories. Free theories serve as a natural check to make sure we are keeping our feet on the ground, but they are solvable by the means we already master. As opposed to the standard approach to quantum field theories, the cMERA treatment of an interacting theory would not involve a perturbative expansion. More than a decade of studies on the lattice, where interacting and noninteracting theories are dealt with on equal grounds, may provide intuition as to how to do the same in the continuum. This hints at the possibility of originating a new, variational approach to quantum field theory which might be relevant in all the fields that use QFT as a tool to understand physical systems.

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