Expressivity of Gaussian tensor networks: a result for fermionic Gaussian MPS

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(joint work w/ J. Ignacio Cirac)

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MAX PLANCK INSTITUTE OF QUANTUM OPTICS

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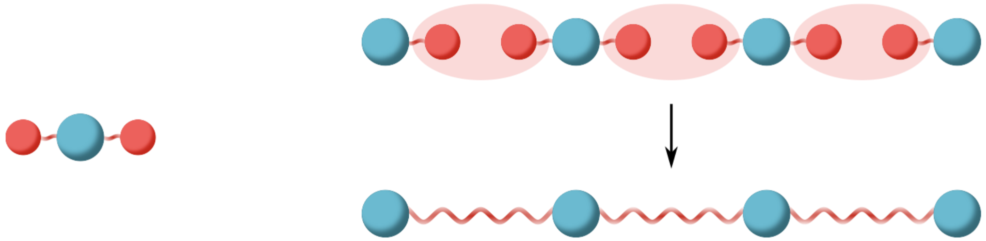
Motivation: Why this talk in this workshop

- Gaussian tensor networks impose the Gaussianity of the global state at the level of the local tensor

[Kraus et al., 09]



- GfPEPS have been used to build examples of topological insulators and superconductors… but they face an obstruction in the form of a no-go theorem:

Any complex vector bundle that is polynomial and analytic is topologically trivial

(Any GfPEPS with a local gapped parent Hamiltonian has vanishing Chern number!)

[Wahl et al., 13]

[Read, Dubail, 13]

Motivation: Why this talk in this workshop

- Gaussian tensor networks being barred from efficiently representing a Gaussian state is pretty demoralizing news…

- We come to wonder about the differences between the interacting and noninteracting approach (*one of the points in the brainstorming session!*)

- This talk contains an example where there is such a separation: Gaussian tensor networks being *strictly worse* (in a particular sense) than non Gaussian ones at representing a Gaussian state

- The context is that of *1d critical systems*, so we will be speaking of Gaussian fermionic matrix product states (GfMPS)

Outline

- Statement of the result
- Ideas of the proof
- Outlook

MPS Approximation Theorem

[Verstraete, Cirac, 06]

Family of states on increasing system sizes

$$|\psi_N\rangle$$

with logarithmic bound on Rényi entropies

$$S_\alpha(\rho_{L,N}) \leq c \log N + c'$$

for

$$S_\alpha(\rho) \equiv \frac{1}{1-\alpha} \log \operatorname{tr} \rho^\alpha$$

$$0 < \alpha < 1$$

gapped and gapless chains

→

Family of MPSs

$$|\psi_N^{MPS}\rangle$$

with *polynomial* bond dimension

$$D(N) \sim \operatorname{poly}(N)$$

and arbitrarily small error for any size

$$\left\| |\psi_N\rangle - |\psi_N^{MPS}\rangle \right\| < \epsilon$$

Desideratum: GfMPS Approximation Theorem

Family of Gaussian fermionic states on increasing system sizes

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???

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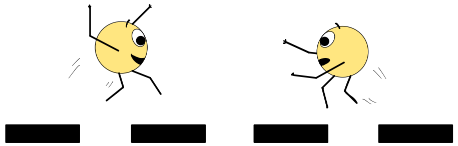
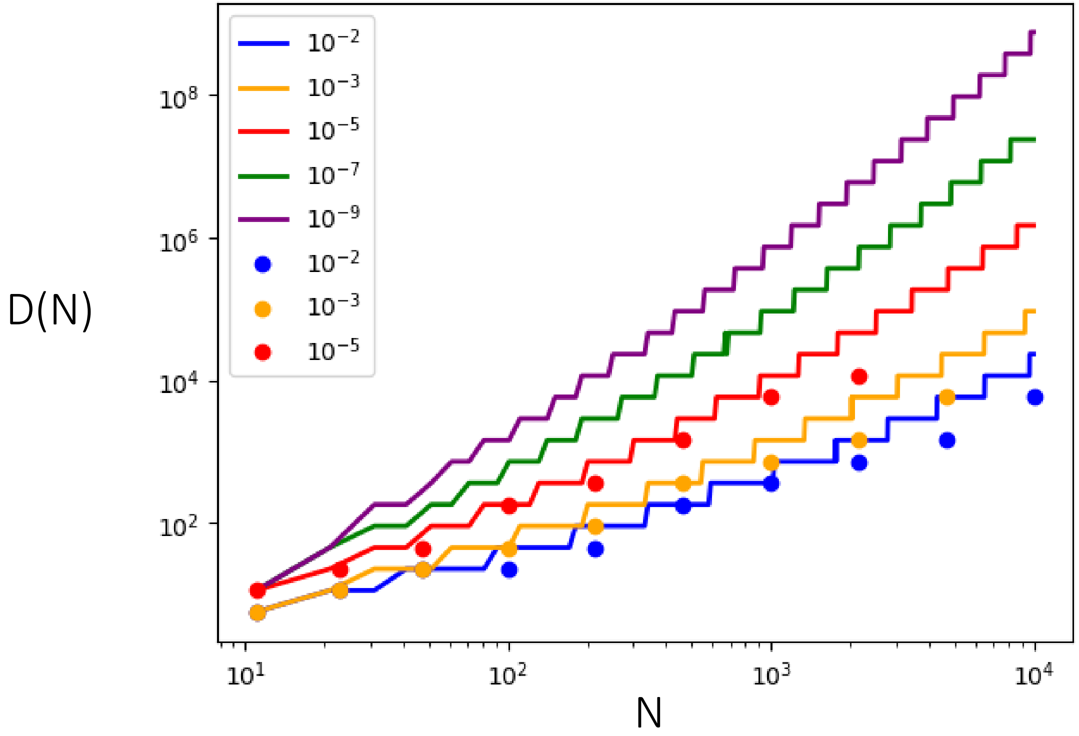
Searching for a Gaussian approximation theorem

Simple noninteracting critical model: hopping fermions



$$H = -\frac{1}{2}\sum_j \left(a_j^\dagger a_{j+1} + a_{j+1}^\dagger a_j\right)$$



Result: our example is a counterexample

[A.F.-R., Cirac., 22]

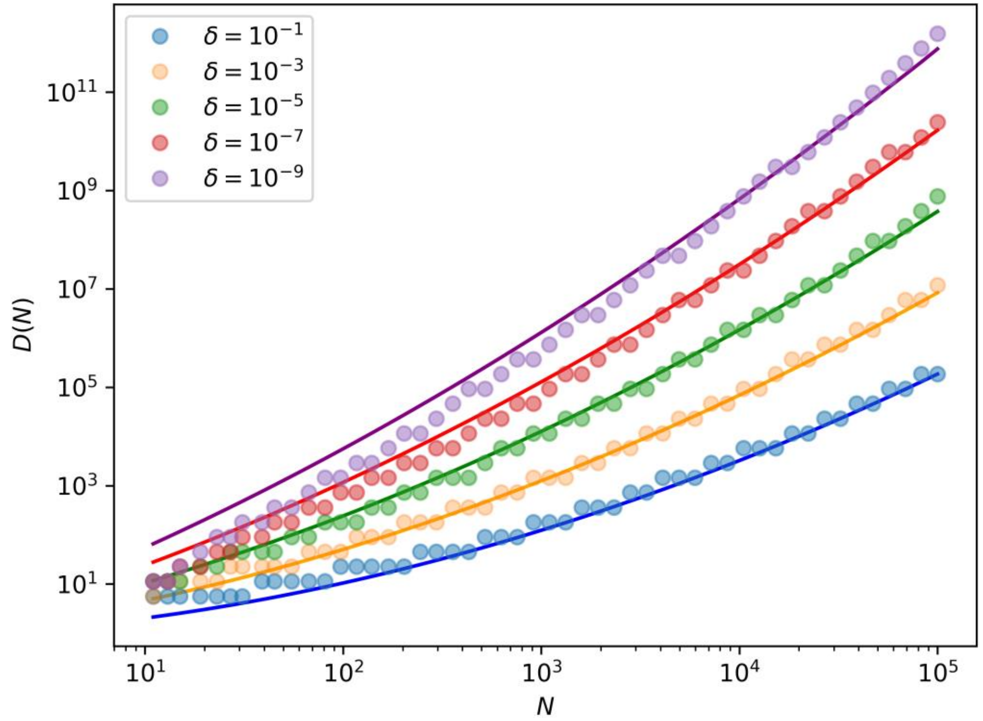
Any Gaussian fermionic MPS approximation to our target ground state with bounded error must have superpolynomial bond dimension

$$D(N) \sim \mathrm{poly}(N) \implies \lim_{N\to\infty} \langle \Psi_N | \Psi_N^{\mathrm{MPS}} \rangle = 0$$

CFT (c = 1 compactified free boson) + heuristic arguments from tensor network folklore

$$D(N,\epsilon) \approx (\eta N)^{\frac{\log 2}{\pi^2} \log\left(\frac{2N \log \eta N}{\pi^2 \epsilon}\right)}$$

Subexponential!



Result: our example is a counterexample

[A.F.-R., Cirac., 22]

Any Gaussian fermionic MPS approximation to our target ground state with bounded error must have superpolynomial bond dimension

$$D(N) \sim \mathrm{poly}(N) \implies \lim_{N\to\infty} \langle \Psi_N | \Psi_N^{\mathrm{MPS}} \rangle = 0$$

However, there exists a (necessarily non Gaussian) MPS approximation to our target ground state with bounded error and polynomial bond dimension!

(since it satisfies the conditions for the MPS approximation theorem)

$$S_\alpha(L) \sim \frac{\alpha+1}{6\alpha} \log L$$

Ideas of the proof: low rank approximation

We bound the error due to the truncation at the central bipartition

Problem: approximate a bipartite state by a state of a given Schmidt rank

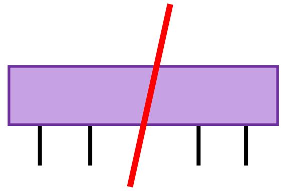
Solution (Eckart-Young-Mirsky thm.): truncate the singular value decomposition (SVD)

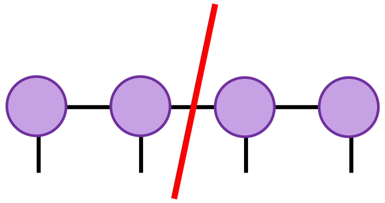
The error is given by the tail of the Schmidt distribution

$$\epsilon = \sum_{j=D+1}^{\infty} s_j^2$$

Ideas of the proof: low rank approximation

We bound the error due to the truncation at the central bipartition





Problem: approximate a Gaussian state by a Gaussian state of a given Schmidt rank

Solution: truncate the Gaussian SVD ("constrained" truncation: whole modes)

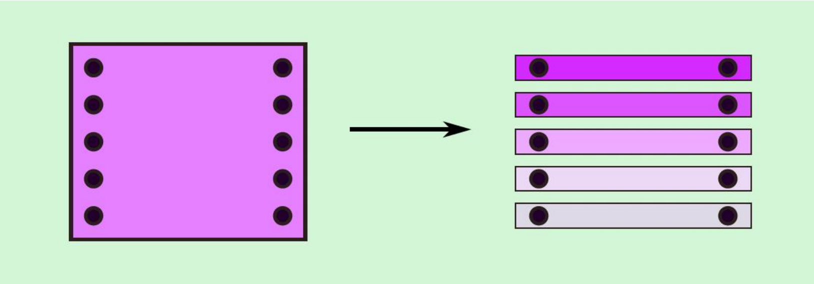
The error is given by the entanglement of the truncated modes

$$\epsilon = 1 - \exp\left(-S_{\infty}^{\mathrm{trunc}}[r]\right)$$

Ideas of the proof: CFT and Toeplitz determinants

We need to estimate the contribution to the entropy of the modes we truncate:

$$\exp\left(-S_{\infty}^{\text{trunc}}[r]\right) = \prod_{i=r+1}^{n} \frac{1+|\lambda_i|}{2}$$



The "Gaussian Schmidt values" come from diagonalizing the correlation matrix.

They can be obtained approximately from CFT (in the continuum limit):

$$|\lambda_j| \approx \tanh\left(\frac{\pi^2}{2}\frac{\varepsilon_j}{\log \ell}\right) \qquad \varepsilon_j \sim j$$

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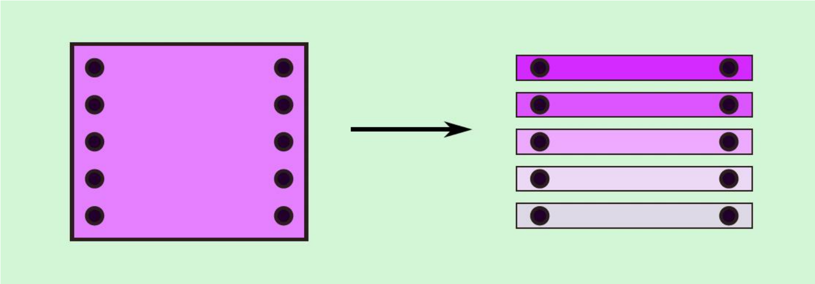


The “Gaussian Schmidt values“ come from diagonalizing the correlation matrix.

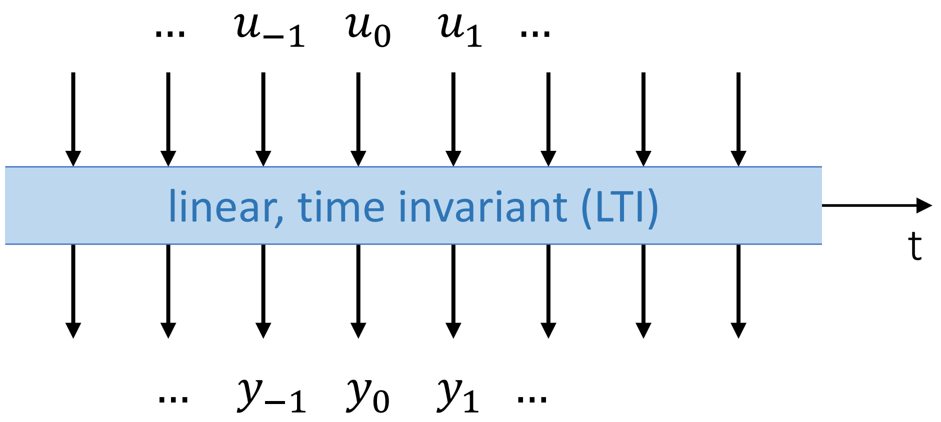
For discrete chains, since it is a Toeplitz matrix , there are asymptotic estimates for ist determinant

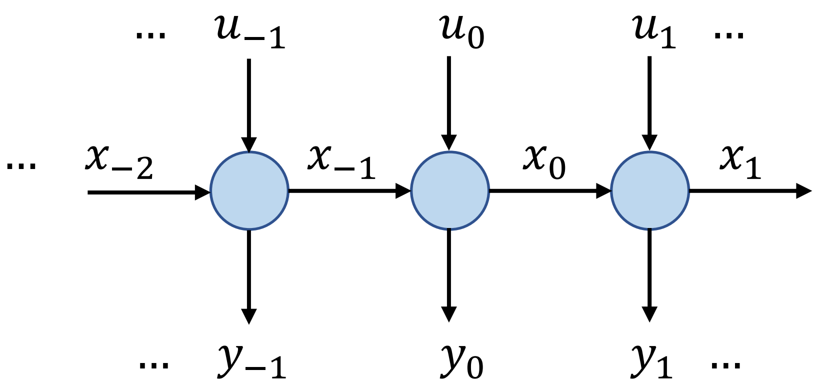
[Jin, Korepin, 04]

[Basor, 79]

$$\Gamma_{ij} = \Gamma_{i-j} \qquad \sum_{i=1}^{r} f(\lambda_i) = \frac{1}{2\pi i}\int_{\mathcal{C}} dz\, f(z)\frac{d}{dz}\log\det\left(zI-\Gamma\right)$$

Fun fact: GfMPS and linear systems





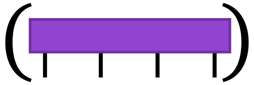
$$U(z) = \sum_k u_k z^{-k}$$

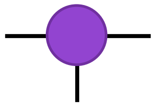
$$Y(z) = \sum_k y_k z^{-k}$$

$$Y(z) = T(z)U(z)$$

$$y_k = Au_k + Bx_{k-1}$$
$$x_k = Cu_k + Dx_{k-1}$$

$$T(z) = A + B(D - zI)^{-1}C$$

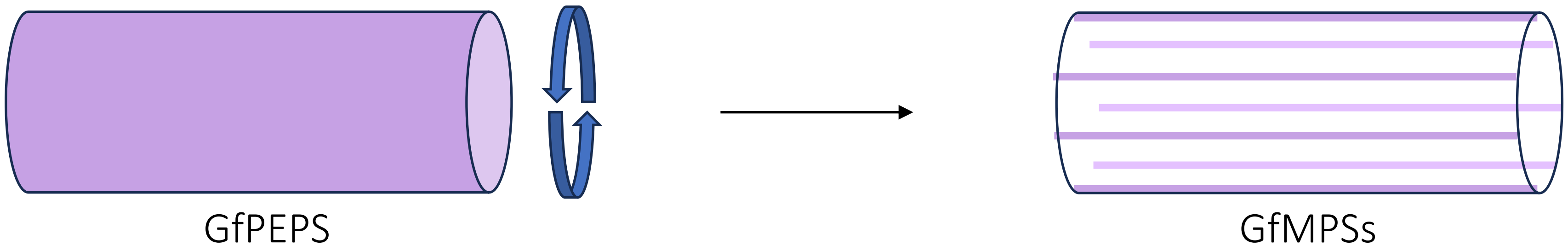
$\Gamma($  $) \sim T(z)$

 $\sim \begin{pmatrix} A & B \\ C & D \end{pmatrix}$

GfMPS with a particular symmetry (time reversal invariance) can be seen as LTI systems and we can import results

Outlook: Higher dimensions?

- Gaussian tensor networks allow for *dimensional reduction*



- Thus, expectedly, Gaussian tensor networks will need superpolynomial bond dimensions to approximate critical states in higher dimensions.

- Approximation with PEPS?

- 2D version of the LTI system correspondence?

Thank you!

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